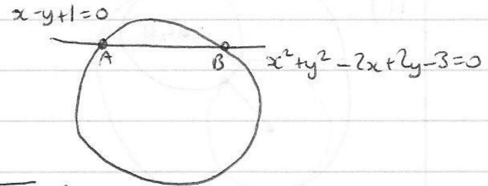
The Circle

Q1.



<u>To find A & B</u>

$x = y - 1$ from eqn of line $\circledast$

Subbing into circle eqn gives:

$\Rightarrow (y-1)^2 + y^2 - 2(y-1) + 2y - 3 = 0$

$y^2 - 2y + 1 + y^2 - \cancel{2y} + 2 + \cancel{2y} - 3 = 0$

$2y^2 - 2y = 0$

$y^2 - y = 0$

$y(y-1) = 0$

$y = 0$ or $y = 1$

$\Rightarrow x = -1$ or $x = 0$ Using $\circledast$

<u>Dist between A & B</u>

$A = (-1, 0) = (x_1, y_1)$ $B = (0, 1) = (x_2, y_2)$

$|AB| = \sqrt{(0+1)^2 + (1-0)^2}$

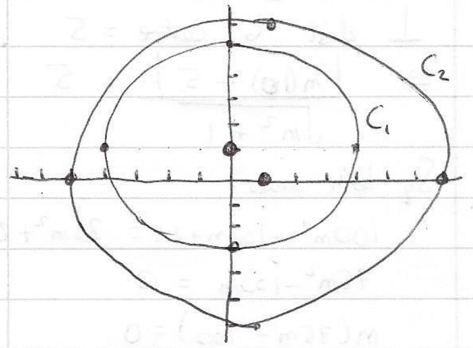
$= \boxed{\sqrt{2}}$

Q2. $x^2 + y^2 - 2y + 1 = 16$: C_1

$\Rightarrow$ Centre $= (0, 1)$ $R = \sqrt{(0)^2 + (1)^2 + 15} = 4$

$x^2 + y^2 - 2x - 35 = 0$: C_2

$\Rightarrow$ Centre $= (1, 0)$ $R = \sqrt{(1)^2 + (0)^2 + 35} = 6$



Q3. To find A & B first:

$2x - y + 6 = 0$

$\Rightarrow y = 2x + 6$ $\circledast$

Sub into circle eqn:

$x^2 + (2x+6)^2 - 2(2x+6) - 9 = 0$

$x^2 + 4x^2 + 24x + 36 - 4x - 12 - 9 = 0$

$5x^2 + 20x + 15 = 0$

$x^2 + 4x + 3 = 0$

$(x+1)(x+3) = 0$

$x = -1$ or $x = -3$

$\Rightarrow y = 4$ or $y = 0$ Using $\circledast$

[AB] is diameter $\Rightarrow$ centre is midpoint of [AB]

$= \left(\frac{-1-3}{2}, \frac{4+0}{2}\right) = (-2, 2)$

Radius $= \frac{1}{2} \times$ length of [AB]

$= \frac{1}{2} \times \sqrt{(-3+1)^2 + (0-4)^2}$

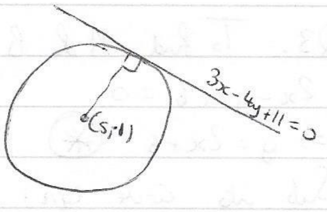
$= \frac{1}{2} \times \sqrt{20}$

$= \sqrt{5}$

$\Rightarrow$ Eqn: $(x+2)^2 + (y-2)^2 = (\sqrt{5})^2$

$\boxed{(x+2)^2 + (y-2)^2 = 5}$

Q4.



i) $\perp$ dist from tangent to centre $= 6$

$$= \frac{|3(5)-4(-1)+11|}{\sqrt{(3)^2+(-4)^2}}$$

$$= \frac{|30|}{5} = \boxed{6}$$

ii) $\perp$ dist from $x+py+1=0$ $= 6$ also

$$\frac{|5+p(-1)+1|}{\sqrt{(1)^2+p^2}} = 6$$

$$\Rightarrow |6-p| = 6\sqrt{p^2+1}$$

Sq both sides:

$$36-12p+p^2 = 36(p^2+1)$$

$$36-12p+p^2 = 36p^2+36$$

$$35p^2+12p = 0$$

$$p(35p+12) = 0$$

$$\boxed{p=0} \text{ or } \boxed{p = \frac{-12}{35}}$$

Q5 Yellow: $x^2+y^2-4x-6y+5=0$

$\Rightarrow$ Centre $=(2,3)$ $R=\sqrt{4+9-5} = \sqrt{8}$

Green: $x^2+y^2-6x-8y+23=0$

$\Rightarrow$ Centre: $(3,4)$ $R=\sqrt{9+16-23} = \sqrt{2}$

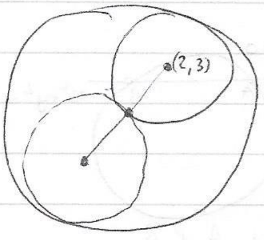
i) Touch internally $\Rightarrow$ dist between centres $= R_{yellow} - R_{green}$

Dist between centres $= \sqrt{(3-2)^2+(4-3)^2}$
$= \sqrt{2}$

$R_{yellow} - R_{green} = \sqrt{8} - \sqrt{2} = \sqrt{2}$

Q.E.D.

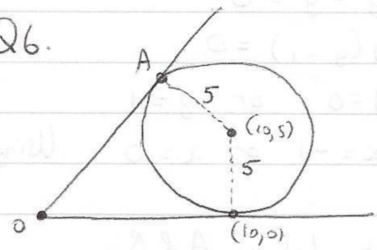
ii)



Diameter of Green $= \sqrt{8} \Rightarrow$ Distance across 2 circles shown $= 2\sqrt{8}$

Diameter of Yellow $= 2\sqrt{8} \Rightarrow$ 2 equal circles would fit.

Q6.



i) Radius of $c = 5$ from diag

$$\Rightarrow (x-10)^2+(y-5)^2 = (5)^2$$

$$x^2-20x+100+y^2-10y+25 = 25$$

$$\boxed{x^2+y^2-20x-10y+100=0}$$

ii) Eqn of any line through $(0,0)$:

$$y-0 = m(x-0)$$

$$y = mx$$

$$mx-y = 0$$

$\perp$ dist to centre $= 5$

$$\Rightarrow \frac{|m(10)-5|}{\sqrt{m^2+1}} = 5$$

Sq both sides:

$$100m^2-100m+25 = 25m^2+25$$

$$75m^2-100m = 0$$

$$m(75m-100) = 0$$

$$m=0 \text{ or } m = \frac{100}{75} = \frac{4}{3}$$

⇒ Subbing back into eqn of tangent

$y = mx$

$y = \frac{4}{3}x$

$3y = 4x$

$\boxed{4x - 3y = 0}$

iii) Find where $4x-3y=0$ intersects circle:

$4x = 3y$

$x = \frac{3y}{4}$

$\Rightarrow \left(\frac{3y}{4}\right)^2 + y^2 - 20\left(\frac{3y}{4}\right) - 10y + 100 = 0$

$\frac{9y^2}{16} + y^2 - 15y - 10y + 100 = 0$

$9y^2 + 16y^2 - 240y - 160y + 1600 = 0$

$25y^2 - 400y + 1600 = 0$

$y^2 - 16y + 64 = 0$

$(y-8)(y-8) = 0$

$\boxed{y = 8}$

$\Rightarrow x = \frac{3(8)}{4} = \boxed{6}$

$\Rightarrow \boxed{A = (6,8)}$

Q7. $(x-3)^2 + (y+4)^2 = 50$

Centre $= (3,-4)$

Radius $= \sqrt{50}$

⊥ dist from tangent to centre = rad

$\frac{|3-(-4)+k|}{\sqrt{(1)^2+(-1)^2}} = \sqrt{50}$

$|k+7| = 10$

$k+7 = 10$ or $k+7 = -10$

$\boxed{k=3}$ $\boxed{k=-17}$

Q8. Radius = ⊥ dist from $3x+y=0$

$= \frac{|3(-2)+(1)|}{\sqrt{(3)^2+(1)^2}}$

$= \frac{|-5|}{\sqrt{10}}$

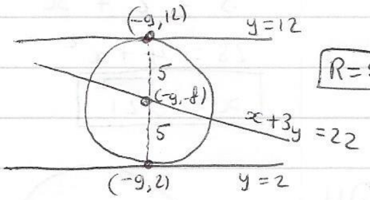
$= \frac{5}{\sqrt{10}}$

Centre $= (-2,1)$ $R = \frac{5}{\sqrt{10}}$

$(x+2)^2 + (y-1)^2 = \left(\frac{5}{\sqrt{10}}\right)^2$

$\boxed{(x+2)^2 + (y-1)^2 = \frac{5}{2}}$

Q9.



$\boxed{R=5}$

$(-g,2)$ on circle

$(-g)^2 + (2)^2 + 2g(-g) + 2f(2) + c = 0$

$g^2 + 4 + 4f - 2g^2 + c = 0$

$-g^2 + 4f + c = -4$ I

$(-g,12)$ on circle

$(-g)^2 + (12)^2 + 2g(-g) + 2f(12) + c = 0$

$-g^2 + 24f + c = -144$ II

$(-g,-f)$ on line $x+3y=22$

$-g - 3f = 22$

$g = -3f - 22$ III

I: $-\cancel{g^2} + 4f + \cancel{c} = -4$

II: $\overset{(+)}{-}g^2 \overset{(-)}{+} 24f \overset{(-)}{+} c = \overset{(+)}{-}144$

$-20f = 140$

$f = -7$

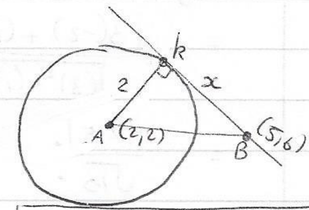
$\Rightarrow g = +21 - 22 = -1$

$\Rightarrow$ Centre $(1,7)$ $R=5$

$\Rightarrow \boxed{(x-1)^2 + (y-7)^2 = 25}$

Q10. $x^2+y^2-4x-4y+4=0$

Centre $=(2,2)$ $\quad R=\sqrt{4+4-4}=2$



$|AB|=\sqrt{(2-5)^2+(2-6)^2}$

$=\sqrt{25}$

$=5$

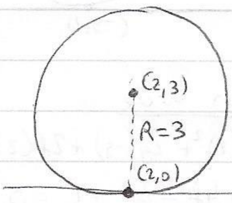
Pyt Thm:

$5^2=2^2+x^2$

$25=4+x^2$

$\boxed{x=\sqrt{21}}$

Q11.



$(x-2)^2+(y-3)^2=9$

$x^2-4x+4+y^2-6y+9=9$

$\boxed{x^2+y^2-4x-6y+4=0}$

Q12. $P: x^2+y^2+2x-2y-23=0$

Cen: $(-1,1)$ $\quad R_1: \sqrt{(-1)^2+(1)^2+23}=5$

$Q: x^2+y^2-14x-2y+41=0$

Cen: $(7,1)$ $\quad R_2: \sqrt{(7)^2+(1)^2-41}=3$

Dist between centres

$=\sqrt{(7+1)^2+(1-1)^2}$

$=8=R_1+R_2$

$\Rightarrow$ Touch Externally

Q13. $x^2+y^2-4x+6y-12=0$

Cen $=(2,-3)$ $\quad R=\sqrt{(2)^2+(-3)^2+12}=5$

Eqn of tangents through $(0,8)$

$\Rightarrow \quad y-8=m(x-0)$

$y-8=mx$

$mx-y+8=0 \quad (*)$

$\perp$ dist to centre $=5$:

$\dfrac{|m(2)-(-3)+8|}{\sqrt{m^2+1}}=5$

$|2m+11|=5\sqrt{m^2+1}$

$4m^2+44m+121=25m^2+25$

$21m^2-44m-96=0$

$(7m-24)(3m+4)=0$

$7m-24=0$ or $3m+4=0$

$m=24/7 \qquad m=-4/3$

$\Rightarrow (*)\quad \frac{24}{7}x-y+8=0$ or $\frac{-4}{3}x-y+8=0$

$\boxed{24x-7y+56=0} \quad \boxed{4x+3y-24=0}$

Q14. $S_1: x^2+y^2=4$

Cen $=(0,0)$ $\quad R_1=\sqrt{4}=2$

$S_2: x^2+y^2-8x-6y+16=0$

Cen $=(4,3)$ $\quad R_2=\sqrt{16+9-16}=3$

Dist between centres $=\sqrt{(4-0)^2+(3-0)^2}$

$=5=R_1+R_2$

$\Rightarrow$ Touch Externally

Common Tgt $=S_1-S_2=0$

$\Rightarrow x^2+y^2-4-(x^2+y^2-8x-6y+16)=0$

$\cancel{x^2}+\cancel{y^2}-4-\cancel{x^2}-\cancel{y^2}+8x+6y-16=0$

$8x+6y-20=0$

$\boxed{4x+3y-10=0}$

Q15. If it's a tangent it must have only one point of contact:

$3x + 4y - 5 = 0$

$3x = 5 - 4y$

$x = \frac{5-4y}{3}$

Sub into circle eqn:

$\left(\frac{5-4y}{3}\right)^2 + y^2 - 6\left(\frac{5-4y}{3}\right) - 8y + 9 = 0$

$\frac{25+16y^2-40y}{9} + y^2 - 10 + \cancel{8y} - \cancel{8y} + 9 = 0$

$25 + 16y^2 - 40y + 9y^2 - 90 + 81 = 0$

$25y^2 - 40y + 16 = 0$

$(5y - 4)(5y - 4) = 0$

$y = 4/5 \Rightarrow x = 3/5$

As there is only 1 point of contact $\Rightarrow 3x + 4y - 5 = 0$ is a tangent.

Q16. $x^2 + y^2 - 6x + 10y + 29 = 0$

$Cen = (3, -5) \quad R = \sqrt{9+25-29} = \sqrt{5}$

Slope of $x - 2y + 5 = 0 = \frac{-1}{-2} = \frac{1}{2}$

$\Rightarrow$ Slope of $\perp = -\frac{2}{1} = -2$

$\Rightarrow$ Eqn of Tgt: $2x + y + d = 0$

$\perp$ dist from Tgt to centre $= \sqrt{5}$

$\frac{|2(3) + (-5) + d|}{\sqrt{(2)^2 + (1)^2}} = \sqrt{5}$

$|d + 1| = 5$

$d + 1 = 5$ or $d + 1 = -5$

$d = 4$ or $d = -6$

$\Rightarrow \boxed{2x + y + 4 = 0}$

$\boxed{2x + y - 6 = 0}$

Q17. $(1, -1)$ on circle:

$\Rightarrow (1)^2 + (-1)^2 + 2g(1) + 2f(-1) + c = 0$

$2g - 2f + c = -2 \quad \text{I}$

$(-6, -2)$ on circle:

$\Rightarrow (-6)^2 + (-2)^2 + 2g(-6) + 2f(-2) + c = 0$

$-12g - 4f + c = -40 \quad \text{II}$

$(3, -5)$ on circle

$\Rightarrow (3)^2 + (-5)^2 + 2g(3) + 2f(-5) + c = 0$

$6g - 10f + c = -34 \quad \text{III}$

Solving I & II

$2g - 2f + \cancel{c} = -2$

$\overset{(+)}{-}12g \overset{(+)}{-}4f \overset{(-)}{+}\cancel{c} = \overset{(+)}{-}40$

$14g + 2f = 38 \quad \text{IV}$

Solving II & III

$-12g - 4f + \cancel{c} = -40$

$(-)\ 6g \overset{(+)}{-}10f \overset{(-)}{+}\cancel{c} = \overset{(+)}{-}34$

$-18g + 6f = -6$

$-6g + 2f = -2 \quad \text{V}$

Solving IV & V

$14g + \cancel{2f} = 38$

$\overset{(+)}{-}6g \overset{(-)}{+}\cancel{2f} = \overset{(+)}{-}2$

$20g = 40$

$\boxed{g = 2}$

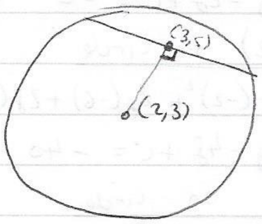
$\Rightarrow \boxed{f = 5}$

$\Rightarrow \boxed{c = 4}$

$\Rightarrow \boxed{x^2 + y^2 + 4x + 10y + 4 = 0}$

Q18. $x^2+y^2-4x-6y-3=0$

Cen $=(2,3)$ $R=\sqrt{4+9+3} = 4$



Slope $= \frac{3-5}{2-3} = \frac{-2}{-1} = 2$

$\Rightarrow$ Slope chord $= -\frac{1}{2}$

$\Rightarrow$ Eqn of chord is:

$y-5 = -\frac{1}{2}(x-3)$

$2y-10 = -x+3$

$\boxed{x+2y-13=0}$

Q19. i) (1,0) on circle

$(1)^2+(0)^2+2g(1)+2f(0)+c=0$

$2g+c=-1$ $\quad$ I

(0,2) on circle

$(0)^2+(2)^2+2g(0)+2f(2)+c=0$

$4f+c=-4$ $\quad$ II

$(-g,-f)$ on line

$-g-3f-11=0$

$g+3f+11=0$ $\quad$ III

Solving I & II

$2g+\cancel{c}=-1$

$(-)4f\overset{(-)}{+}\cancel{c} = \overset{(+)}{-}4$

$2g-4f=3$ $\quad$ IV

Solving III & IV

III $\times 2$: $\cancel{2g}+6f=-22$

IV : $(-)\cancel{2g} \overset{(+)}{-} 4f = \overset{(-)}{}3$

$10f=-25$

$f=\frac{-25}{10}=\frac{-5}{2}$

$\Rightarrow$ $g=-\frac{7}{2}$

$\Rightarrow$ $c=6$

$\Rightarrow$ $\boxed{x^2+y^2-7x-5y+6=0}$

ii) Origin in, on or outside?

$(0)^2+(0)^2-7(0)+5(0)+6=0$

$6>0$

$\Rightarrow$ Outside. Q.E.D.

Q20. $x^2+y^2-10kx+6y+60=0$

i) Cen: $(5k,-3)$ $R=\sqrt{25k^2+9-60}=\sqrt{25k^2-51}$

ii) Radius $=7$

$\Rightarrow \sqrt{25k^2-51}=7$

$25k^2-51=49$

$25k^2=100$

$k^2=4$

$k=\pm 2$

As $k>0 \Rightarrow \boxed{k=2}$

Q21. (2,4) on circle:

$\Rightarrow (2)^2+(4)^2+2g(2)+2f(4)+c=0$

$4g+8f+c=-20$ I

(-6,0) on circle:

$\Rightarrow (-6)^2+(0)^2+2g(-6)+2f(0)+c=0$

$-12g+c=-36$ II

Radius $=\sqrt{40}$

$\Rightarrow \sqrt{g^2+f^2-c}=\sqrt{40}$

$g^2+f^2-c=40$ III

Solving I & II

$4g+8f+\cancel{c}=-20$

$\overset{(+)}{-}12g \overset{(-)}{} \overset{(-)}{\cancel{+c}} = \overset{(+)}{-}36$

$16g+8f=16$

$2g+f=2$

$\Rightarrow f=2-2g$ IV

Putting II and IV into III

$g^2+(2-2g)^2-(12g-36)=40$

$g^2+4g^2-8g+4-12g+36-40=0$

$5g^2-20g=0$

$g^2-4g=0$

$g(g-4)=0$

$g=0$ or $g=4$

$\Rightarrow f=2$ or $f=-6$

$\Rightarrow c=-36$ or $c=12$

$\Rightarrow \boxed{x^2+y^2+4y-36=0}$ or $\boxed{x^2+y^2+8x-12y+12=0}$

Q22. Any line $\perp$ to $3x-4y+1=0$ will have equation $4x+3y+d=0$

$\perp$ dist to centre = radius:

centre $=(4,-1)$ $R=\sqrt{16+1+8}=5$

$\dfrac{|4(4)+3(-1)+d|}{\sqrt{(4)^2+(3)^2}}=5$

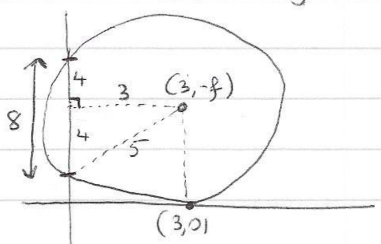
$|13+d|=25$

$13+d=25$ or $13+d=-25$

$d=12$ or $d=-38$

$\Rightarrow \boxed{4x+3y+12=0 \text{ or } 4x+3y-38=0}$

Q23. Sketch of diagram:



Line from centre to a chord, $\perp$ to a chord, bisects the chord

$\Rightarrow$ from diag: radius = 5 using Pythagoras Thm with 3 and 4.

Radius = 5 $\Rightarrow$ y coord of centre must be 5

$\Rightarrow$ Cen (3, 5) $R=5$

$(x-3)^2+(y-5)^2=25$

$x^2-6x+9+y^2-10y+\cancel{25}-\cancel{25}=0$

$\boxed{x^2+y^2-6x-10y+9=0}$