Formal Proofs For LCHL:

a) Proof by contradiction that $\sqrt{2}$ is irrational:

Assume $\sqrt{2}$ is not irrational => $\sqrt{2}$ can be written in the form $\frac{a}{b}$

=> $\sqrt{2} = \frac{a}{b}$ (where a and b have no common factor)

=> $2 = \frac{a^2}{b^2}$

=> $2b^2 = a^2$

As $2b^2$ is an even number => a^2 must be even

=> 'a' can be written in the form 2k

=> $2b^2 = (2k)^2$

=> $2b^2 = 4k^2$

=> $b^2 = 2k^2$, which means b is even as well.

If 'a' and 'b' are even, then 2 must divide into both => Contradiction

=> $\sqrt{2}$ is irrational

b) Example of Proof by contradiction in Geometry:

* Assume opposite is true and prove that the opposite is impossible.

- Example 1:
  Prove that an equilateral Δ is also an acute-angled Δ (i.e. has no angle bigger than 90°)

  Proof:
  Assume opposite true i.e. equilateral Δ is NOT acute-angled
  $\Rightarrow$ it has one angle bigger than 90°

  As Δ is equilateral, all angles are equal
  $\Rightarrow$ it has 3 angles bigger than 90°
  But...
  Sum of 3 angles $= 180^\circ$ so this is impossible!

c) Derivation of Sum of Infinite Geometric Series Formula:

- Consider the Geometric Series a + ar + ar^2 + ar^3 + ...
- The sum of the first term, sum of the first two terms, sum of the first three terms etc.

  $S_1 = a$

  $S_2 = a + ar$

  $S_3 = a + ar + ar^2$

  .

  .

  $S_n = a + ar + ar^2 + \ldots\ldots + ar^{n-1}$
- We can now find the limit of this series as n approaches infinity:

$$\lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$$

- As the $\frac{a}{1-r}$ is a constant, we can simply move that out in front of the limit:

$$\lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}\lim_{n\to\infty}(1-r^n)$$

- As long as r is between -1 and 1 i.e. $|r| < 1$, then the limit of r^n as n approaches infinity will be 0.
- So, the limit simplifies to:

$$\frac{a}{1-r}(1-0) = \frac{a}{1-r}$$

d) Proof of DeMoivre's Theorem by Induction:

To Prove: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

Step 1: $(\cos\theta + i\sin\theta)^1 = \cos n\theta + i\sin n\theta$, which is true.

Step 2: $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

Step 3: $(\cos\theta + i\sin\theta)^{k+1} = \cos(k+1)\theta + i\sin(k+1)\theta$

LHS:

$(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k.(\cos\theta + i\sin\theta)^1$ (Using 1st Law of Indices)

- We know $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$ from P(k), so we can replace that in the line above giving:

$= (\cos k\theta + i\sin k\theta).(\cos\theta + i\sin\theta)$ (using P(k))

- And then multiplying out the brackets above gives:

$= \cos k\theta\cos\theta + \cos k\theta i\sin\theta + i\sin k\theta\cos\theta + i^2\sin\theta\sin k\theta$

$= \cos k\theta\cos\theta + \cos k\theta\sin\theta\, i + \sin k\theta\cos\theta\, i - \sin\theta\sin k\theta$

- Now collect Real and Imaginary parts together:

$= (\cos k\theta\cos\theta - \sin\theta\sin k\theta) + i[\cos k\theta\sin\theta + \cos\theta\sin k\theta]$

- Now we use the identities cos(A+B) = cosAcosB – sinAsinB and sin(A+B) = cosAsinB + sinAcosB to simplify the expressions in brackets:

$= (\cos(k\theta + \theta) + i\sin(k\theta + \theta))$

- Finally, factorise out the θ:

$= (\cos(k+1)\theta + i\sin(k+1)\theta)$, which is = RHS above.

=> $P(k+1)$ *is true, if* $P(k)$ *is true*

=> $P(n)$ *is true* $\forall\, n \in N$. Q.E.D.

e) Derivation of Amortisation Formula:

Proof:

- Let P = the amount borrowed, A = the repayment amount, t = the time period of repayment and i = the interest rate.
- The amount borrowed has to be equal to the present value of all the repayments:

$$\Rightarrow P = \frac{A}{(1+i)^1} + \frac{A}{(1+i)^2} + \frac{A}{(1+i)^3} + \cdots + \frac{A}{(1+i)^t}$$

- A Geometric Series with a = $\frac{A}{(1+i)}$ and r = $\frac{1}{(1+i)}$, so:

$$\Rightarrow P = \frac{\frac{A}{(1+i)}\left(1 - \left(\frac{1}{(1+i)}\right)^t\right)}{1 - \frac{1}{(1+i)}} \qquad \text{(Using Sn formula } \frac{a(1-r^n)}{1-r}\text{)}$$

$$\Rightarrow P = \frac{\left(\frac{A}{(1+i)}\right)\left(1 - \frac{1^t}{(1+i)^t}\right)}{1 - \frac{1}{(1+i)}}$$

$$\Rightarrow P = \frac{\left(\frac{A}{(1+i)}\right)\left(1 - \frac{1^t}{(1+i)^t}\right)}{\frac{1(1+i)-1}{(1+i)}}$$

$$\Rightarrow P = \frac{\left(\frac{A}{(1+i)}\right)\left(\frac{1(1+i)^t - 1}{(1+i)^t}\right)}{\frac{i}{(1+i)}}$$

$$\Rightarrow P = \left(\frac{A}{(1+i)}\right)\left(\frac{(1+i)^t - 1}{(1+i)^t}\right) \times \frac{(1+i)}{i}$$

$$\Rightarrow P = \left(\frac{A}{\cancel{(1+i)}}\right)\left(\frac{(1+i)^t - 1}{(1+i)^t}\right) \times \frac{\cancel{(1+i)}}{i}$$

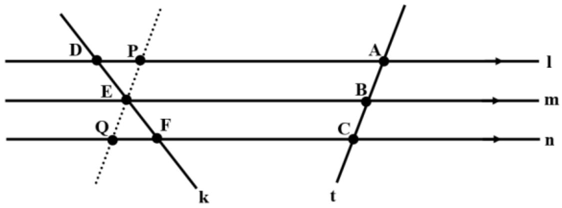
$$\Rightarrow P = (A)\left(\frac{(1+i)^t - 1}{i(1+i)^t}\right)$$

$$\Rightarrow A = \frac{P(i(1+i)^t)}{(1+i)^t - 1} \qquad \text{Q.E.D.}$$

f) Theorem 11:

Theorem 11: If three parallel lines cut off equal segments on some transversal line, then they will cut off equal segments on any other transversal.

Diagram:



Given: Three parallel lines l, m and n, intersecting the transversal t at the points A, B and C such that |AB| = |BC|. Another transversal k intersects the lines at D, E and F.

To Prove: |DE| = |EF|

Construction: Through E, construct a line parallel to t and intersecting l at the point P and n at the point Q.

Proof: PEBA and EQCB are parallelograms (from construction)

=> |PE| = |AB| and |EQ| = |BC| (opposite sides of a parallelogram)

But |AB| = |BC| (Given)

So => |PE| = |EQ|.

Consider now △DEP and △FEQ:

|PE| = |EQ| (from above)

|∠PED| = |∠FEQ| (Vertically opposite angles)

|∠DPE| = |∠FQE| (Alternate angles)

=> △DEP and △FEQ are congruent (ASA)

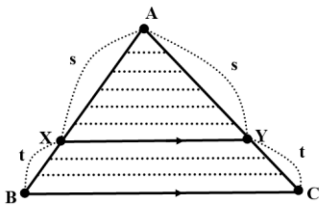
=> |DE| = |EF|.

Q.E.D.

g) Theorem 12:

Theorem 12: Let ABC be a triangle. If a line XY is parallel to BC and cuts [AB] in the ratio s : t, then it also cuts [AC] in the same ratio.

Diagram:



Given: The triangle ABC with XY parallel to BC.

To Prove: $\frac{|AX|}{|XB|} = \frac{|AY|}{|YC|}$

Construction: Divide [AX] into s equal parts and [XB] into t equal parts. Draw a line parallel to BC through each point of division.

Proof: The parallel lines make intercepts of equal length along the line [AC] (From Theoerem 11)

=> [AY] is divided into s equal intercepts and [YC] is divided into t equal intercepts.

=> $\frac{|AY|}{|YC|} = \frac{s}{t}$

But $\frac{|AX|}{|XB|} = \frac{s}{t}$

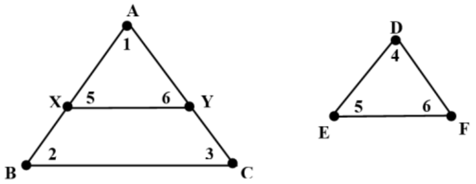
=> $\frac{|AX|}{|XB|} = \frac{|AY|}{|YC|}$

Q.E.D.

h) Theorem 13:

Theorem 13: If two triangles ABC and DEF are similar, then their sides are proportional in order.

Diagram:



Given: Similar triangles ABC and DEF

To prove: $\frac{|AB|}{|DE|} = \frac{|BC|}{|EF|} = \frac{|AC|}{|DF|}$

Construction: Mark the point X on [AB] such that |AX| = |DE|.
Mark the point Y on [AC] such that |AY| =|DF|.
Join XY.

Proof: $\triangle$ AXY and $\triangle$ DEF are congruent. (SAS)

=> $|\angle AXY| = |\angle DEF| = |\angle 5|$ (Corresponding angles)

=> $|\angle AXY| = |\angle ABC|$ (as triangles ABC and DEF are similar)

=> $XY \parallel BC$ (as angles 2 and 5 are corresponding)

=> $\frac{|AB|}{|AX|} = \frac{|AC|}{|AY|}$ (A line parallel to one side divides other side in the same ratio….Theorem 12)

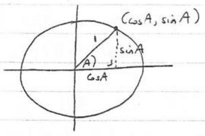
=> $\frac{|AB|}{|DE|} = \frac{|AC|}{|DF|}$

Similarly, it can be proven that $\frac{|AB|}{|DE|} = \frac{|BC|}{|EF|}$

=> $\frac{|AB|}{|DE|} = \frac{|BC|}{|EF|} = \frac{|AC|}{|DF|}$ Q.E.D.

i) Trig Identities 1 - 9:

1) $\cos^2 A + \sin^2 A = 1$:



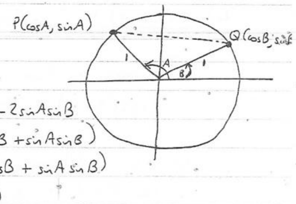
Using Pythagoras Thm

$1^2 = (\cos A)^2 + (\sin A)^2$

$\Rightarrow \boxed{\cos^2 A + \sin^2 A = 1}$

2) $\cos(A-B) = \cos A \cos B + \sin A \sin B$:

Let $P(\cos A, \sin A)$ and $Q(\cos B, \sin B)$ be two points on a unit circle.



Using distance formula:

$|PQ| = \sqrt{(\cos A - \cos B)^2 + (\sin A - \sin B)^2}$

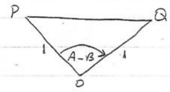
$|PQ|^2 = \cos^2 A + \cos^2 B - 2\cos A \cos B + \sin^2 A + \sin^2 B - 2\sin A \sin B$

$= (\cos^2 A + \sin^2 A) + (\cos^2 B + \sin^2 B) - 2(\cos A \cos B + \sin A \sin B)$

$= 1 + 1 - 2(\cos A \cos B + \sin A \sin B)$

$= 2 - 2(\cos A \cos B + \sin A \sin B)$

Using cosine rule to find PQ



$|PQ|^2 = (1)^2 + (1)^2 - 2(1)(1)\cos(A-B)$

$= 2 - 2\cos(A-B)$

Equating our 2 expressions for $|PQ|^2$:

$2 - 2\cos(A-B) = 2 - 2(\cos A \cos B + \sin A \sin B)$ (Take 2 from both sides)

$-2\cos(A-B) = -2(\cos A \cos B + \sin A \sin B)$ ($\div$ both sides by -2)

$\boxed{\cos(A-B) = \cos A \cos B + \sin A \sin B}$

3) $\cos(A+B) = \cos A \cos B - \sin A \sin B$:

We know from (2) that

$\cos(A - B) = \cos A \cos B + \sin A \sin B$

If we fill in $-B$ instead of B:

$\cos(A-(-B)) = \cos A \cos(-B) + \sin A \sin(-B)$

$\Rightarrow \cos(A+B) = \cos A \cos(-B) + \sin A \sin(-B)$

Since $\cos(-B) = \cos B$ and $\sin(-B) = -\sin B$

$\Rightarrow \boxed{\cos(A+B) = \cos A \cos B - \sin A \sin B}$

4) $\sin(A+B) = \sin A \cos B + \cos A \sin B$:

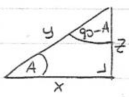
We know from (2) that

$\cos(A-B) = \cos A \cos B + \sin A \sin B$

If we fill in $90-A$ instead of A:

$\cos((90-A)-B) = \cos(90-A)\cos B + \sin(90-A)\sin B$

Note:



$\sin(90-A) = x/y = \cos A$

and $\cos(90-A) = z/y = \sin A$

$\Rightarrow \cos((90-A)-B) = \sin A \cos B + \cos A \sin B$

$\Rightarrow \cos(90-(A+B)) = \sin A \cos B + \cos A \sin B$

$\Rightarrow \boxed{\sin(A+B) = \sin A \cos B + \cos A \sin B}$

5) $\sin(A-B) = \sin A \cos B - \cos A \sin B$:

We know from (4) that

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

If we fill in $-B$ instead of B:

$\sin(A+(-B)) = \sin A \cos(-B) + \cos A \sin(-B)$

$\sin(A-B) = \sin A \cos(-B) + \cos A \sin(-B)$

Since $\cos(-B) = \cos B$ and $\sin(-B) = -\sin B$

$\Rightarrow \boxed{\sin(A-B) = \sin A \cos B - \cos A \sin B}$

6) $\cos 2A = \cos^2 A - \sin^2 A$:

We know from (3) that

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

If we replace B by A we get:

$\cos(A+A) = \cos A \cos A - \sin A \sin A$

$\Rightarrow \boxed{\cos 2A = \cos^2 A - \sin^2 A}$

7) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$:

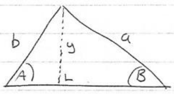
$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A\cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B} \quad \text{using (3) and (4)}$$

We now divide each of the 4 terms by $\cos A \cos B$

$$\Rightarrow \tan(A+B) = \frac{\dfrac{\sin A \cancel{\cos B}}{\cos A \cancel{\cos B}} + \dfrac{\cancel{\cos A}\sin B}{\cancel{\cos A}\cos B}}{\dfrac{\cancel{\cos A}\cancel{\cos B}}{\cancel{\cos A}\cancel{\cos B}} - \dfrac{\sin A \sin B}{\cos A \cos B}}$$

$$= \boxed{\frac{\tan A + \tan B}{1 - \tan A \tan B}}$$

8) Sine Rule $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$:



Angle A can be acute or obtuse so we have 2 cases.

A is Acute:

$\sin A = y/b \qquad \sin B = y/a$

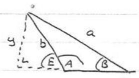
$\Rightarrow b\sin A = y \qquad a\sin B = y$

$\Rightarrow b\sin A = a\sin B$

$\Rightarrow \dfrac{b\sin A}{\sin B} = a$

$\Rightarrow \dfrac{\sin A}{\sin B} = \dfrac{a}{b}$

A is Obtuse:



$E = 180 - A$

$\sin E = \sin(180 - A)$

$= \sin A$

$\sin E = y/b \qquad \sin B = y/a$

$\Rightarrow \sin A = y/b \qquad \Rightarrow a\sin B = y$

$\Rightarrow b\sin A = y$

$\Rightarrow b\sin A = a\sin B$

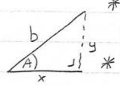
$\Rightarrow \dfrac{\sin A}{\sin B} = \dfrac{a}{b}$

Similarly, we can show $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$

$$\Rightarrow \boxed{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}}$$

9) Cosine Rule $a^2 = b^2 + c^2 - 2bc\cos A$:

Consider the Δ below:

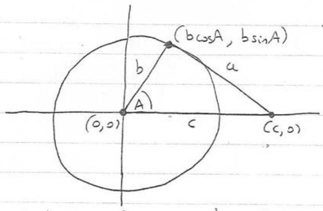


$*\ \cos A = x/b$

$\Rightarrow x = b\cos A$

$*\ \sin A = y/b$

$\Rightarrow y = b\sin A$



Using distance formula to find a :

$$a = \sqrt{(b\cos A - c)^2 + (b\sin A - 0)^2}$$

$$\Rightarrow a^2 = b^2\cos^2 A + c^2 - 2bc\cos A + b^2\sin^2 A$$

$$= b^2(\cos^2 A + \sin^2 A) + c^2 - 2bc\cos A$$

$$= b^2(1) + c^2 - 2bc\cos A$$

$$\Rightarrow \boxed{a^2 = b^2 + c^2 - 2bc\cos A}$$