Topic 4: Algebra

1) The Basics:

a) Adding / Subtracting Algebraic Expressions:

Notes:

- We can only add / subtract 'like terms'.
- 'Like terms' are terms that have the same letter part or the same variables
  e.g. 5d and -2d are 'like terms' but 5d and $5d^2$ are NOT 'like terms'

Example 1:

$4a + 5 + 2a - 3$
$= 6a + 2$

Example 2:

$3x^2y - 4y^2 - x^2y - 3y + 2y^2$
$= 2x^2y - 2y^2 - 3y$

b) Multiplying Expressions:

Notes:

- when multiplying we follow the order Signs, Numbers, Letters
- When multiplying the letters together we must remember the first law of indices.....$a^m \times a^n = a^{m+n}$ i.e. Add the Powers

Example 1: Multiplying Terms

$4a^2 \times 2a^5$ (Multiply signs...(+).(+) = +)
$= 8a^7$ (Multiply Numbers & Add Powers)

Example 2: Removing Brackets

$2(g + 4)$
$= 2g + 8$

Example 3: Removing Brackets

$(2x - 3)(x + 2)$ ("Split and Repeat")
$= 2x(x + 2) - 3(x + 2)$
$= 2x^2 + 4x - 3x - 6$
$= 2x^2 + x - 6$

c) Dividing Expressions:

Tip: Can we factorise the numerator or the denominator?

Example:

$$\frac{2x+6}{x^2-9} = \frac{2(x+3)}{(x+3)(x-3)} = \frac{2}{x-3}$$

2) Algebraic Fractions:

a) Adding/Subtracting Fractions:

Note:

- When adding/subtracting fractions together we find the common denominator and bring both terms up to the same denominator first.

Example 1:

$$\frac{x+3}{5} - \frac{2x-1}{3}$$
$$= \frac{3(x+3)}{15} - \frac{5(2x-1)}{15}$$
$$= \frac{3x+9}{15} - \frac{10x-5}{15}$$
$$= \frac{3x+9-(10x-5)}{15}$$
$$= \frac{3x+9-10x+5}{15}$$
$$= \frac{-7x+14}{15}$$

Example 2:

$$\frac{5}{x+3} - \frac{2}{x-4}$$
$$= \frac{5(x-4)}{(x+3)(x-4)} - \frac{2(x+3)}{(x+3)(x-4)}$$
$$= \frac{5(x-4)-2(x+3)}{(x+3)(x-4)}$$
$$= \frac{5x-20-2x-6}{(x+3)(x-4)}$$
$$= \frac{3x-26}{(x+3)(x-4)}$$

Note: A shortcut we can use when doing the type of questions above:

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}$$

b) Long Division:

Note: We can also divide algebraic fractions using long division. Always check for factorising first though as it can simplify the question a lot. When doing long division questions remember: "Daddy, Mammy, Sister Brother", which stands for Divide, Multiply, Subtract, Bring Down

Example:

Simplify $\dfrac{x^3 - 5x^2 + 10x - 12}{x - 3}$.

$$\begin{array}{r l}
 & x^2 - 2x + 4 \\
x - 3 \,\big)\!\!\! & \overline{x^3 - 5x^2 + 10x - 12} \\
- & \underline{x^3 \stackrel{+}{-} 3x^2} \\
 & -2x^2 + 10x \\
 & \underline{\stackrel{+}{-}2x^2 \stackrel{-}{+} 6x} \\
 & 4x - 12 \\
- & \underline{4x \stackrel{+}{-} 12} \\
 & 0
\end{array}$$

3) Factorising and Manipulation of Formulae:

a) Factorising:

1. Taking out the HCF (taking out what's common)

e.g.s

i) $2x - 10$
$= 2(x - 5)$

ii) $3x^2 - 18x$
$= 3x(x - 6)$

2. Grouping (always has 4 terms)

e.g.s

i) $ax + ay + bx + by$
$= a(x + y) + b(x + y)$
$= (x + y)(a + b)$

ii) $3p - 3q - pk + kq$
$= 3(p - q) - k(p - q)$
$= (p - q)(3 - k)$

3. Quadratic (always has 3 terms x^2, x, a)

e.g.s

i) $x^2 + 5x + 6$
$= (x + 3)(x + 2)$

ii) $x^2 - 3x - 18$
$= (x - 6)(x + 3)$

4. Difference of 2 Squares (always 2 terms with a '-' between)

Note: Watch for square numbers: 1, 4, 9, 16, 25, 36, 49, 64, 81….

e.g.s

i) $x^2 - 9y^2$
$= (x)^2 - (3y)^2$
$= (x - 3y)(x + 3y)$

ii) $16a^2 - 25b^2$
$= (4a)^2 - (5b)^2$
$= (4a - 5b)(4a + 5b)$

b) Manipulation of Formulae:

Steps:

1) Get rid of any brackets, fractions or square roots.
2) Bring all terms with the letter you want to the LHS and move everything else to the RHS.
3) Factorise out the letter you want (if necessary).
4) Divide both sides to leave the letter you want on the LHS.

Example: Write r, in terms of p and q.

$$\sqrt{\frac{p}{r-q}} = p$$

$\Rightarrow \left(\sqrt{\frac{p}{r-q}}\right)^2 = (p)^2$ (Squaring both sides to get rid of $\sqrt{\ }$)

$\Rightarrow \frac{p}{r-q} = p^2$

$\Rightarrow p = p^2(r - q)$ (Multiplying both sides by (r - q))

$\Rightarrow p = p^2r - p^2q$

$\Rightarrow -p^2r = -p - p^2q$ (Bringing term with r to LHS)

$\Rightarrow p^2r = p + p^2q$ (Changing all the signs)

$\Rightarrow r = \frac{p + p^2q}{p^2}$ (dividing both sides by p^2)

4) Solving Equations:

a) Solving Linear Equations: (x only)

Steps:

1. Remove all brackets and any fractions
2. Bring all terms with an 'x' to one side and numbers to the other side
3. Tidy up both sides by putting together 'like terms'.
4. Solve the simple equation remaining.

Example: $2(x - 3) = 4(x + 1)$

$2x - 6 = 4x + 4$
$2x - 4x = 4 + 6$
$-2x = 10$
$x = \frac{10}{-2}$
$\Rightarrow x = -5$

b) Solving Linear Equations With Fractions:

Tip:

"Kill" all fractions first by multiplying all terms by something that ALL denominators divide into.

Example: Solve $\frac{2x-3}{4} + \frac{x+6}{5} = \frac{3}{2}$

In this case 20 will kill the fractions, so multiply across by 20:

$$\cancel{20}\left(\frac{2x - 3}{\cancel{4}}\right) + \cancel{20}\left(\frac{x + 6}{\cancel{5}}\right) = \cancel{20}\left(\frac{3}{\cancel{2}}\right)$$

$5(2x - 3) + 4(x + 6) = 10(3)$
$10x - 15 + 4x + 24 = 30$
$10x + 4x = 30 + 15 - 24$
$14x = 21$
$\Rightarrow x = \frac{21}{14} = \frac{3}{2}$

c) Solving Quadratic Eqns by factorising: (Equations with an x^2)

Steps:

1. Bring all terms to the left-hand side (LHS) and leave '0' on the RHS
2. Factorise the LHS (See section on Factorising in previous tab)
3. If LHS can't be factorised the 'Quadratic Formula' needs to be used (See Example 3 on the right)
4. Let each factor be = 0
5. Solve the two simple equations to find the two answers.

Example 1: $x^2 - 3x - 18 = 0$

$(x - 6)(x + 3) = 0$
$x - 6 = 0$ or $x + 3 = 0$
$\Rightarrow x = 6$ or $x = -3$

Example 2: $4x^2 - 25 = 0$

$(2x - 5)(2x + 5) = 0$
$2x - 5 = 0$ or $2x + 5 = 0$
$\Rightarrow 2x = 5$ or $2x = -5$
$\Rightarrow x = \frac{5}{2}$ or $x = \frac{-5}{2}$

d) Solving Quadratic Eqns using the "-b Formula":

Note: This method can be used for ALL quadratic equations.

If $ax^2 + bx + c = 0$ is a quadratic equation, then the roots of the equation are given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

See Tables pg 20

Example 3: Solve $x^2 - 2x - 5 = 0$.

In this case: a = 1, b = -2 and c = -5

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-5)}}{2(1)}$$

$$\Rightarrow x = \frac{2 \pm \sqrt{24}}{2}$$

$$\Rightarrow x = 3.45 \quad or \quad x = -1.45$$

e) Quadratic Eqns with fractions:

Example: Solve $\frac{2}{x+1} - \frac{3}{x-2} = \frac{5}{2}$

Method 1: (Multiply across by common denominator)
In this case the common denominator would be $2(x+1)(x-2)$:

$$2(\cancel{x+1})(x-2)\frac{2}{\cancel{x+1}} - 2(x+1)(\cancel{x-2})\frac{3}{\cancel{x-2}} = \cancel{2}(x+1)(x-2)\frac{5}{\cancel{2}}$$

$2(x - 2)(2) - 2(x + 1)(3) = 5(x + 1)(x - 2)$
$4x - 8 - 6x - 6 = 5x^2 - 5x - 10$
$-5x^2 + 3x - 4 = 0$
$5x^2 - 3x + 4 = 0$…….and solve this as before.

Method 2: (Tidy up both sides into single fractions and cross multiply) (See Section 2 – Example 2)

$$\frac{2}{x+1} - \frac{3}{x-2} = \frac{5}{2}$$
$$\frac{2(x-2) - 3(x+1)}{(x+1)(x-2)} = \frac{5}{2}$$
$$\frac{-x-7}{(x+1)(x-2)} = \frac{5}{2}$$
$$2(-x-7) = 5(x+1)(x-2)$$
$$-2x - 14 = 5(x^2 - x - 2)$$
$$-2x - 14 = 5x^2 - 5x - 10$$
$$5x^2 - 3x + 4 = 0 \quad \text{etc.}$$

f) Forming Quadratic Equation from the roots:

Method 1:

Steps:
1. Let x = both of the roots.
2. Create two factors that are = 0.
3. Multiply the two factors together using "split and repeat".

Example: Find the quadratic equation with roots -1 and 3.

$$x = -1 \quad or \ x = 3$$
$$x + 1 = 0 \quad or \ x - 3 = 0$$
$$(x+1)(x-3) = 0$$
$$x(x-3) + 1(x-3) = 0$$
$$x^2 - 3x + x - 3 = 0$$
$$x^2 - 2x - 3 = 0$$

Method 2: Use the formula

Need to know to use this method.

$$x^2 - (sum\ of\ roots)x + (product\ of\ roots) = 0$$

Example: Find the quadratic equation with roots -1 and 3.

$$x^2 - (sum\ of\ roots)x + (product\ of\ roots) = 0$$
$$x^2 - (-1+3)x + ((-1)(3)) = 0$$
$$x^2 - 2x - 3 = 0$$

5) Simultaneous Equations:

a) Two Linear Equations:

Steps:
1. Choose a variable to eliminate e.g. 'y'
2. Multiply one or both equations to make no. in front of y the same
3. Multiply the 2nd equation by -1, if necessary, to make signs in front of 'y' different.
4. Add the two equations to eliminate 'y' and solve for 'x'.
5. Put x back into one of the equations to find y.

Example: Solve the equations below:

A: $2x - 3y = 7$
B: $3x + 2y = 4$

$Ax2$: $4x - \cancel{6y} = 14$ (mult by 2 to get 6 in front of y)
$Bx3$: $9x + \cancel{6y} = 12$ (mult by 3 to get 6 in front of y)

$13x = 26$ (adding both equations together)
$\Rightarrow x = \frac{26}{13}$ (dividing both sides by 13)
$\Rightarrow x = 2$

Putting x into A:

A: $2x - 3y = 7$
$\Rightarrow 2(2) - 3y = 7$
$\Rightarrow 4 - 3y = 7$
$\Rightarrow -3y = 7 - 4$
$\Rightarrow -3y = 3$
$\Rightarrow y = \frac{3}{-3}$ (dividing both sides by -3)
$\Rightarrow y = -1$

b) One Linear, One Quadratic:

Steps:
1. Use the linear equation to get one variable on its own.
2. Sub this into the quadratic equation.
3. Multiply out and solve the resulting quadratic equation.
4. Sub your two values back into the expression from step 1.

Example: Solve the equations $x - y = 2$ and $2x^2 + y^2 = 36$.

L: $x - y = 2$
C: $2x^2 + y^2 = 36$

Step 1: Use the linear equation to get one variable on its own:

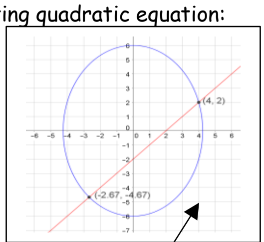
L: $x - y = 2$
=> $x = y + 2$ *

Step 2: Substitute our expression for x into equation C:

C: $2x^2 + y^2 = 36$
=> $2(y + 2)^2 + y^2 = 36$

Step 3: Multiply out and solve the resulting quadratic equation:

=> $2(y^2 + 4y + 4) + y^2 = 36$
=> $2y^2 + 8y + 8 + y^2 - 36 = 0$
=> $3y^2 + 8y - 28 = 0$
$(3y + 14)(y - 2) = 0$
=> $3y + 14 = 0$ OR $y - 2 = 0$
$3y = -14$ OR $y = 2$
$y = \frac{-14}{3}$



Step 4: Sub your two values back into the expression from step 1:

$x = y + 2$ *

When $y = 2$
$x = 2 + 2$
$x = 4$

When $y = \frac{-14}{3}$
$x = \frac{-14}{3} + 2$
$x = \frac{-8}{3}$

So, our two solutions are: (4, 2) and $\left(\frac{-8}{3}, \frac{-14}{3}\right)$.

6) Inequalities:

a) Solving Inequalities:

Notes:

- Need to know the types of numbers (See Arithmetic 1b)
- Same rules as solving linear equations (See Algebra 4a)
- One difference: if you have to multiply/divide both sides of an inequality by a **NEGATIVE** number, we must **CHANGE THE DIRECTION** of the inequality.

Example 1: Graph the solution to $3 - 4x < 11,\ x \in Z$.

$3 - 4x < 11$

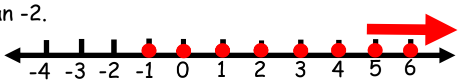
$-4x < 11 - 3$

$-4x < 8$

$\frac{-4x}{-4} < \frac{8}{-4}$ (dividing both sides by -4)

$x > -2$ (Note sign change because divided by -4)

For the number line, we're looking for all the Integers that are bigger than -2.



Example 2: Graph the solution to $3(x - 2) \le -3,\ x \in R$.

$3(x - 2) \le -3$

$3x - 6 \le -3$

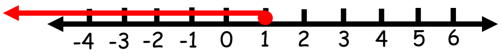
$3x \le -3 + 6$ (adding 6 to both sides)

$3x \le 3$

$\frac{3x}{3} \le \frac{3}{3}$ (dividing both sides by 3)

=> $x \le 1$

For the number line, we're looking for all the Real numbers that are smaller than or equal to 1.



b) Double Inequalities:

Method 1: (Break inequality up into two inequalities)

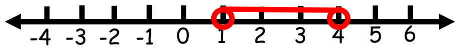
Example: Graph the solution to $-4 < 3x - 7 < 5,\ x \in R$.

We break up the inequality into two inequalities as indicated by the red and blue below.

$-4 < 3x - 7 < 5$

Inequality 1	Inequality 2
$-4 < 3x - 7$	$3x - 7 < 5$
$-4 < 3x - 7$	$3x < 5 + 7$
$3 < 3x$	$3x < 12$
$1 < x$	$x < 4$

So, combining our two solutions, we want all the Real numbers that are bigger than 1 but less than 4. (not including the 1 and the 4)



Method 2:

Tip:

Do the same thing to all three parts of the inequality to leave an 'x' in the middle.

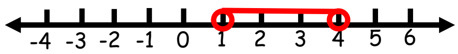
Example: Graph the solution to $-4 < 3x - 7 < 5,\ x \in R$.

$-4 + 7 < 3x \cancel{-7} \cancel{+7} < 5 + 7,\ x \in R$ (+ 7 to eliminate -7 in the middle)

$3 < 3x < 12$

$\frac{3}{3} < \frac{3x}{3} < \frac{12}{3}$ (dividing all parts by 3)

$1 < x < 4$



7) Word Problems:

Tips:

1. Read the question a couple of times before attempting it.
2. Underline any Mathematical key words e.g. sum, product, total.
3. Let 'x' be what you are looking for, if there is one unknown. Use 'x' and 'y' for two unknowns.
4. Form an equation.
5. Solve the equation.
6. If you are unable to form an equation, try using "trial and improvement" to solve the problem. You need to show all trials and workings.
7. Check your answer(s).

Example 1: Find two consecutive natural numbers whose sum is 83.

- Keywords: consecutive, natural and sum
- Let x = 1st number, so that means x + 1 = 2nd number
- Their sum is 83 ('sum' means they add to 83)

  => x + x + 1 = 83 (equation formed)

  => 2x + 1 = 83

  => 2x = 83 – 1

  => 2x = 82

  => x = 41 (dividing both sides by 2)

  => second number is x + 1 = 42
- Check.....41 + 42 = 83

Example 2: A shop sells 50 sofas in a week. A leather sofa costs €1000 and a fabric sofa costs €750. The shop sells €42,500 worth of sofas. How many of each type are sold?

- Let x = no. of leather sofas and y = no. of fabric sofas
- Total number of sofas = 50

  => x + y = 50 (first equation formed)
- Total money = €42,500

  => 1000x + 750y = 42500 (second equation formed)
- Can solve the 2 simultaneous equations now to find x and y. (See Section 5a)