Topic 9: Calculus (Integration)

1) The Basics:

a) Integrating Expressions:

Notes:

- Symbol: $\int$
- To integrate a constant w.r.t x, just multiply the constant by x
- In general, to integrate, we:

Add 1 to the power and divide by the new power.

$$\int \frac{1}{x} dx = \ln x + c$$

- If there are no limits of integration, then you have to add a constant of integration 'c'.

Examples:

i) $\int (4x^3 + 3)\, dx = 4\left(\frac{x^{3+1}}{4}\right) + 3\left(\frac{x^{0+1}}{1}\right) = x^4 + 3x + c$

ii) $\int \frac{1}{x^3} dx = \int x^{-3}\, dx = \frac{x^{-2}}{-2} = \frac{1}{-2x^2} + c$

b) Definite Integrals:

Notes:

- When you know the limits, you fill them in after integrating the expression.
- If F(x) is the integral of f(x) then:

$$\int_a^b f(x).dx = F(b) - F(a)$$

Example:

$\int_{-2}^{2} x^2 + 4x\, dx = \frac{x^3}{3} + 2x^2$

$\frac{(2)^3}{3} + 2(2)^2 - [\frac{(-2)^3}{3} + 2(-2)^2]$ (filling in the limits)

$= \frac{16}{3}$

2) Trig/Exponential Functions:

a) Trigonometric Functions:

Notes:

- To integrate we use the Tables pg 26:

$f(x)$	$\int f(x).dx$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\tan x$	$\ln\lvert \sec x \rvert$

- Based on the ones given to us above, we also have:

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax + c$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax + c$$

Example:

$\int \sin 5x\, dx = -\frac{1}{5}\cos 5x + c$

b) Exponential Functions:

Notes:

- To integrate we use the Tables pg 26:

$f(x)$	$\int f(x).dx$
e^x	e^x
e^{ax}	$\frac{1}{a}e^{ax}$
a^x	$\frac{a^x}{\ln a}$

Examples:

i) $\int e^{3x}\, dx = \frac{1}{3}e^{3x} + c$

ii) $\int e^{-4x}\, dx = -\frac{1}{4}e^{-4x} + c$

iii) $\int 3^x\, dx = \frac{3^x}{\ln 3} + c$

3) Average Value of a Function:

Notes:

- To find the average value of a function between two limits a and b:

$$Ave\ Value = \frac{1}{b-a}\int_a^b f(x).dx$$

Example: Find the average value of the function $k(t) = 2t^2 - 3t + 5$ on the interval [0,3].

$= \frac{1}{3-(0)}\int_0^3 2t^2 - 3t + 5.dt$

$= \frac{1}{3}[\frac{2t^3}{3} - \frac{3t^2}{2} + 5t\ ].dt$

$= \frac{2t^3}{9} - \frac{t^2}{2} + \frac{5t}{3}$

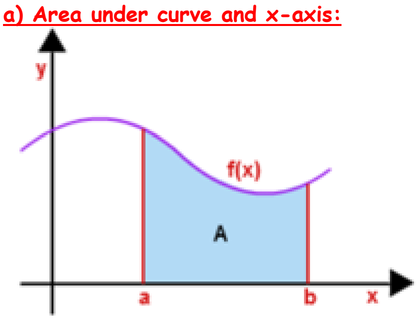
Putting in the limits:

$= \frac{2(3)^3}{9} - \frac{(3)^2}{2} + \frac{5(3)}{3} - (\frac{2(0)^3}{9} - \frac{(0)^2}{2} + \frac{5(0)}{3})$

$= \frac{11}{2}$

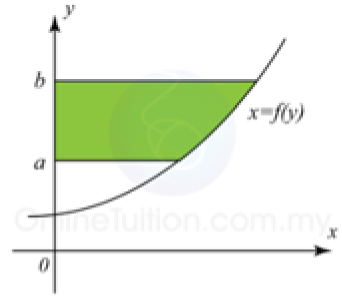
4) Area Under/Between Curves:

a) Area under curve and x-axis:



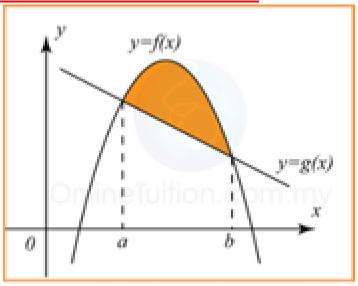
$$\text{Area of Shaded} = \int_a^b y \, .dx$$

b) Area between curve and y-axis:



$$\text{Area of Shaded} = \int_a^b x \, .dy$$

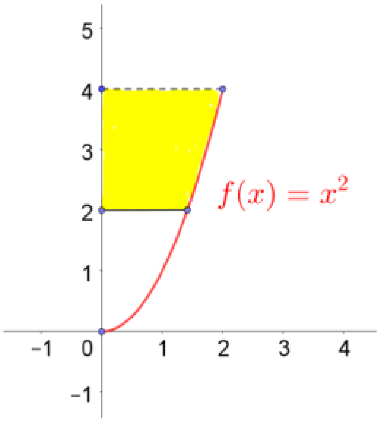
c) Area between two curves:



Area of Shaded

$$= \int_a^b f(x) \, .dx - \int_a^b g(x) \, .dx$$

Example 1: A graph of the function $f(x) = x^2$ is shown below in the domain $0 \le x \le 2$. Find the area bounded by the section of the graph shown, and the y-axis.



- When finding the area between a curve and the y-axis we have to start by writing the function in terms of y:

$$y = x^2$$

$$\Rightarrow x = \sqrt{y} = y^{\frac{1}{2}}$$

- The area of the yellow section now is given by:

$$= \int_2^4 y^{\frac{1}{2}} \, .dy$$

- Integrating the function gives:

$$\int_2^4 y^{\frac{1}{2}} \, .dy = \frac{y^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2y^{\frac{3}{2}}}{3}$$

- We now evaluate the integral between the two limits:

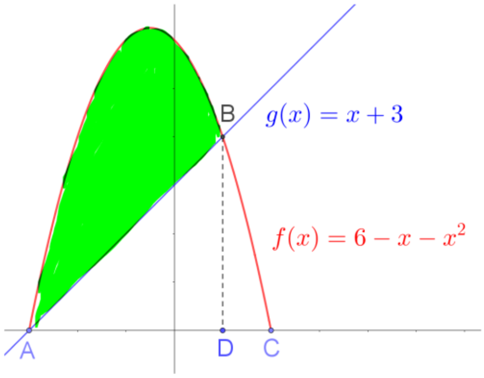
Area $= \frac{2y^{\frac{3}{2}}}{3}$ (between 2 and 4)

$$= \frac{2(4)^{\frac{3}{2}}}{3} - \frac{2(2)^{\frac{3}{2}}}{3}$$

$$= \frac{16}{3} - 1.89$$

= 3.44units^2

Example 2: A graph of the functions $f(x) = 6 - x - x^2$ and $g(x) = x + 3$. Find the area of the shaded region.



- Firstly, find the coordinates of A, B and C.
- To find A and C, we need to solve where $f(x) = 0$:

  $6 - x - x^2 = 0$

  => $x^2 + x - 6 = 0$ (multiplying across by -1)

  => $(x + 3)(x - 2) = 0$

  => $x = -3$ or $x = 2$ => $A = (-3,0)$ and $C = (2,0)$
- Now, find B by, using the method from Pg 17 Topic 8(b):

  L: y = x + 3   C: y = 6 - x - x^2
- Put our expression for y from equation L into equation C:

  C: x + 3 = 6 - x - x^2

  => x^2 + 2x - 3 = 0

  => (x + 3)(x - 1) = 0

  => x = -3 or x = 1 => $A = (-3,0)$ and $B = (1,0)$
- We can now update our diagram with the relevant x-values.
- The area of the shaded region can now be calculated by calculating the area under f(x) between -3 and 1 and then subtracting the area of triangle ADB. ($\frac{1}{2}$ x base x height)
- We can also use integration to find the area of the triangle ADB.
- So, the shaded area will be:

$$= \int_{-3}^{1} 6 - x - x^2 \, .dx - \int_{-3}^{1} x + 3 \, .dx$$

$$= 6x - \frac{x^2}{2} - \frac{x^3}{3} - \left(\frac{x^2}{2} + 3x\right)$$

- Evaluating this integral within the limits gives the area of the shaded region:

$$= \frac{56}{3} - 8 = \frac{32}{3} units^2$$