- **Chapter 6: Impacts and Collisions**
- **Topic 29: Impacts**

- If you threw a basketball straight up in the air and allowed it to bounce freely, you would have noticed that the height of each bounce gets progressively smaller, until it stops bouncing altogether.
- If the ball falls with a speed of 15m/s, and then rebounds off the ground with a speed of 12m/s, it has retained only $\frac{4}{5}$ of its speed.
- We say that the coefficient of restitution, e, between the basketball and the ground is $\frac{4}{5}$ or 0.8.
- To introduce some vectors into the bouncing ball situation, if the ball was falling at 15m/s, it could be represented by $-15\vec{j}$ m/s.
- When the ball rebounds, it is travelling at $12\vec{j}$ m/s.
- So,

  The ratio $\frac{12}{-16} = -\frac{3}{4} = -e$
- In general,

$$\boxed{-e = \frac{new\ velocity}{old\ velocity} = \frac{v}{u}}$$

- If the collision is perfectly elastic i.e. the ball retained all of it's speed after impact, then e = 1.
- If the collision is perfectly inelastic i.e. the ball retained none of its speed after impact, then e = 0.

  => 'e' can only be between 0 and 1.

- Example 1: Pg 109 Ex 6A Q5

A ball of mass 0.1 kg is dropped from rest at a height 22.5 m above horizontal ground. If the coefficient of restitution is $\frac{5}{7}$, find:

i) its speed just before impact
ii) its speed just after impact
iii) the impulse imparted to the ball
iv) the loss of kinetic energy.

Solution:

i) We can use our equations of motion from Chap 2 to solve the first part:

u = 0, s = 22.5, a = g, v = ?

$v^2 = u^2 + 2as$

=> $v^2 = (0)^2 + 2(g)(22.5)$

=> $v^2 = 441$

=> $v = \sqrt{441} = 21$ m/s

ii) As the ball is falling down, its velocity just before impact will be $-21\vec{j}$ m/s.

- We will let p = the velocity after impact

$$-e = \frac{new\ velocity}{old\ velocity} = \frac{p}{-21} = -\frac{5}{7}$$

=> $\frac{p}{21} = \frac{5}{7}$

=> $7p = 105$

=> $p = \frac{105}{7} = 15$

=> the new velocity will be $15\vec{j}$ m/s, so its speed will be 15 m/s.

Note: A quicker way is to just get $\frac{5}{7}$ of its speed, and change the sign because of the change of direction.

iii) The impulse imparted will be:

$\vec{I} = m\vec{v} - m\vec{u}$

=> $\vec{I} = (0.1)15\vec{j} - (0.1)(-21)\vec{j}$

=> $\vec{I} = 1.5\vec{j} + 2.1\vec{j}$

=> $\vec{I} = 3.6\vec{j}$ Ns.

What would impulse imparted to the ground be?

iv) We finally calculate the kinetic energy before and after the impact, to find the change in kinetic energy:

$KE_{before} = \frac{1}{2}mv^2$

=> $KE_{before} = \frac{1}{2}(0.1)(21)^2$

=> $KE_{before} = 22.05 Joules$

$KE_{after} = \frac{1}{2}mv^2$

$KE_{after} = \frac{1}{2}(0.1)(15)^2$

$KE_{after} = 11.25 Joules$

=> Loss in KE = 22.05 - 11.25

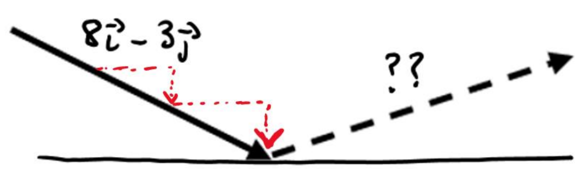
= 10.8J.

Note: What about if the ball struck the ground at an angle instead??

- Example 2: A ball of mass 2 kg, moving at $8\vec{i} - 3\vec{j}$, strikes the ground. What is its velocity after the impact, if e = $\frac{2}{3}$? Find also the loss in KE.

  Solution:

- The ball's motion is shown below:



- Firstly, the impact only takes place in the $\vec{j}$ direction, so velocity in the $\vec{i}$ direction is unaffected.
- So, the velocity after the collision will be: $8\vec{i} + \frac{2}{3}(3)\vec{j} = 8\vec{i} + 2\vec{j}$.

Note this for bouncing projectiles later on.

Ask what would happen if bounce took place off vertical surface.

- We're going to look at calculating KE loss using two methods here to illustrate a very useful shortcut:

Method 1: Using $\vec{\imath}$ and $\vec{\jmath}$ components	Method 2: Using $\vec{\jmath}$ components only
- For this method, we have to find the magnitude of the velocities before and after impact first: Initial Velocity = $\lvert\vec{u}\rvert = \lvert 8\vec{\imath} - 3\vec{\jmath}\rvert$ $= \sqrt{(8)^2 + (-3)^2}$ $= \sqrt{64 + 9}$ $= \sqrt{73}$ Final Velocity = $\lvert\vec{v}\rvert = \lvert 8\vec{\imath} + 2\vec{\jmath}\rvert$ $= \sqrt{(8)^2 + (2)^2}$ $= \sqrt{64 + 4}$ $= \sqrt{68}$ - So: $KE_{before} = \frac{1}{2}mu^2 = \frac{1}{2}(2)\left(\sqrt{73}\right)^2 = 73$ $KE_{after} = \frac{1}{2}mv^2 = \frac{1}{2}(2)\left(\sqrt{68}\right)^2 = 68$ => Loss = 73 – 68 = 5 J	- For this method, we can ignore the $\vec{\imath}$ components as they are unchanged in the impact - So: $KE_{before} = \frac{1}{2}mu^2 = \frac{1}{2}(2)(-3)^2 = 9$ $KE_{after} = \frac{1}{2}mv^2 = \frac{1}{2}(2)(2)^2 = 4$ => Loss = 9 – 4 = 5 J

Important Note:

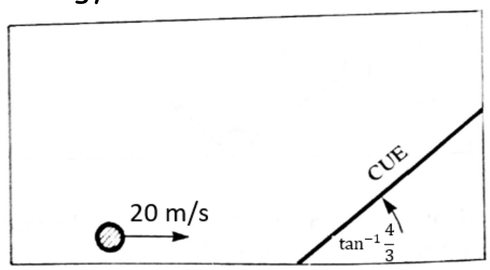
- We can use Method 2 above any time that we're asked for "Kinetic Energy Loss".
- If, however, we are asked "Percentage Energy Lost", we have to take extra care as we need to put the energy lost over the TOTAL energy in the system before the collision.
- To calculate the TOTAL energy in the system before the collision, we would need to use both $\vec{\imath}$ and $\vec{\jmath}$ components as in Method 1.

Classwork Questions: Pg 109/110 Ex 7A Qs 1/4/6/7/9

- Example 3: Pg 110 Ex 6A Q13

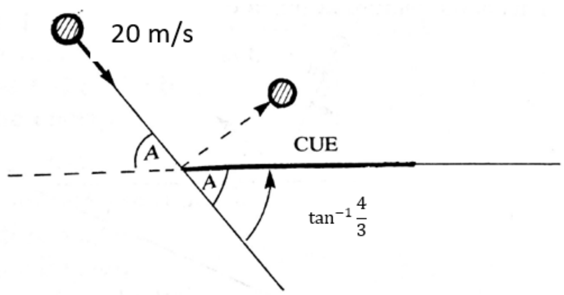
  A ball of mass m moves horizontally with speed 20 m/s towards a smooth barrier xy which makes an angle $\tan^{-1}(\frac{4}{3})$ with the horizontal. The coefficient of restitution is 0.75. Find:

  i) the speed of the ball after the impact
  ii) the magnitude of the impulse imparted to the ball by the impact.
  iii) the loss in kinetic energy



Solution:

i) We make the line of the cue our x-axis, so with rotation of the cue into a horizontal position, the picture now looks like this:



- As $\tan A = \frac{4}{3}$ => $\sin A = \frac{4}{5}$ and $\cos A = \frac{3}{5}$, so the velocity of the ball before impact will be:

$$= 20\cos A\,\vec{\imath} - 20\sin A\,\vec{\jmath}$$
$$= 20(\frac{3}{5})\vec{\imath} - 20(\frac{4}{5})\vec{\jmath}$$
$$= 12\vec{\imath} - 16\vec{\jmath} \text{ m/s}$$

- Again, the impact takes place along the $\vec{\jmath}$ direction only
- The $\vec{\imath}$ velocity will remain unchanged by the impact, so we can ignore it.
- Again, we will let p represent the velocity after impact:

$$-e = \frac{new\ velocity}{old\ velocity} = \frac{p}{-16} = -\frac{3}{4}$$
$$=> \frac{p}{16} = \frac{3}{4}$$
$$=> 4p = 48$$
$$=> p = 12$$

=> the new velocity will be $12\vec{\imath} + 12\vec{\jmath}$ m/s and the new speed will be $\sqrt{(12)^2 + (12)^2} = \sqrt{144 + 144} = \sqrt{288} = 12\sqrt{2}$ or 16.97 m/s.

ii) The impulse imparted will be: (again ignoring the $\vec{\imath}$ components)

$$\vec{I} = m\vec{v} - m\vec{u}$$

=> $\vec{I} = (m)(12\vec{\jmath}) - (m)(16\vec{\jmath})$

=> $\vec{I} = 12m\vec{\jmath} - 16m\vec{\jmath}$

=> $\vec{I} = -4m\vec{\jmath}$ Ns

=> the magnitude of the impulse is 4m Ns.

iii) As in the last example, we can use our shortcut and just use the $\vec{\jmath}$ components as we are only being asked for KE loss:

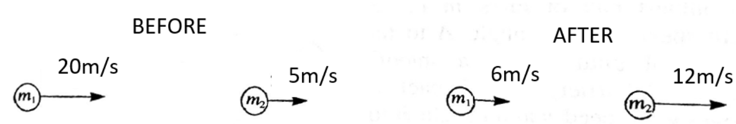
$KE_{before} = \frac{1}{2}mu^2$	$KE_{after} = \frac{1}{2}mv^2$
=> $KE_{before} = \frac{1}{2}(m)(-16)^2$	$KE_{after} = \frac{1}{2}(m)(12)^2$
=> $KE_{before} = 128m\ Joules$	$KE_{after} = 72m\ Joules$

=> Loss in KE = 128m – 72m

= 56m J.

Classwork Questions: Pg 109/110 Ex 6A Qs 10/11/14/15

- Topic 30: Direct Collisions
  - In the previous topic, particles were bouncing off stationary objects like the ground, or the cushions on a snooker table.
  - In this topic, we look at particles that collide with other moving particles.
  - Imagine two smooth metal spheres moving at $20\vec{\imath}$ m/s and $5\vec{\imath}$ m/s, and after they collide, they're travelling with speeds of $6\vec{\imath}$ m/s and $12\vec{\imath}$ m/s, the question is, what is the coefficient of restitution in this case?



  - To solve this problem, we use something we learned in chapter 4, Relative Velocity.
  - If you imagine being a flea on the sphere m_2, before the collision, you wouldn't be able to feel your own sphere moving, but you would see another sphere approaching……but at what speed?
    - The speed would be the relative speed of $20\vec{\imath}$ - $5\vec{\imath}$ = $15\vec{\imath}$ m/s. (As $\vec{V}ab = \vec{V}\text{a} - \vec{V}\text{b}$)
  - After the impact, you would see the other sphere moving away from you
    - The speed would be the relative speed of $6\vec{\imath}$ - $12\vec{\imath}$ = $-6\vec{\imath}$ m/s.
  - So, you would have seen a particle arriving at $15\vec{\imath}$ m/s and then leaving at $-6\vec{\imath}$ m/s, so:

$$-e = \frac{new\ velocity}{old\ velocity} = \frac{-6}{15} = -\frac{2}{5}$$

=> e = $\frac{2}{5}$

- In general, if the spheres have velocities of $u_1\vec{\imath}$,$u_2\vec{\imath}$ before impact and $v_1\vec{\imath}$, $v_2\vec{\imath}$ after impact, then: $\boxed{-e = \frac{v_1 - v_2}{u_1 - u_2}}$ → Newton's Experimental Law of Restitution
- And the other rule that still applies in these situations is the Principle of Conservation of Momentum: $\boxed{m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2}$

- Example 1: Pg 112 Ex 6B Q5

  Two elastic spheres of mass 5 kg and 3 kg, travelling in opposite directions, collide directly. The speeds before collision are 6 m/s and 4 m/s respectively. If the coefficient of restitution between the spheres is $\frac{1}{3}$, calculate:

  i) the speed of each sphere after the collision

  ii) the loss in KE suffered by the 5 kg sphere due to the collision

  iii) the change in momentum of the 3 kg sphere

  N.B. "Opposite directions"

  Solution:

  i) We will let p = the velocity of the first ball after impact and q = the velocity of the second ball after impact, and so our "Before, Mass and After" table will be:

Before	Mass	After
$6\vec{\imath}$	5	$p\vec{\imath}$
$-4\vec{\imath}$	3	$q\vec{\imath}$

- We will now use our two rules from above:

Newton's Law of Restitution (NLR)	Law of Conservation of Momentum (LCM)
$-e = \frac{v_1 - v_2}{u_1 - u_2}$ => $-e = \frac{p-q}{6-(-4)} = -\frac{1}{3}$ => $\frac{p-q}{10} = -\frac{1}{3}$ => $3(p-q) = -10$ => $3p - 3q = -10$……..Eqn 1	$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$ => $(5)(6) + (3)(-4) = (5)(p) + (3)(q)$ => $30 - 12 = 5p + 3q$ => $5p + 3q = 18$………Eqn 2

We can now solve equations 1 and 2 together:

Eqn 1: $3p - 3q = -10$

Eqn 2: $\underline{5p + 3q = 18}$

=> 8p = 8

=> p = 1

Subbing p into equation 1 gives:

$3(1) - 3q = -10$

$3 - 3q = -10$

$3q = 13$

$q = \frac{13}{3}$

=> velocities after impact are $1\vec{\imath}$ m/s and $\frac{13}{3}\vec{\imath}$ m/s, so the speeds are 1 m/s and $\frac{13}{3}$ m/s

ii) The kinetic energy loss of the 5kg sphere will be:

$KE_{before} = \frac{1}{2}(5)(6)^2$ $KE_{after} = \frac{1}{2}(5)(1)^2$

$KE_{before} = \frac{1}{2}(5)(36)$ $KE_{after} = \frac{1}{2}(5)(1)$

$KE_{before} = 90$ $KE_{after} = \frac{5}{2} = 2.5$

- So, the loss in Kinetic Energy is:

90 - 2.5 = 87.5 J.

iii) The change in momentum (or impulse) imparted to the 3 kg spheres will be:

$\vec{I} = m_2\vec{v}_2 - m_2\vec{u}_2$

=> $\vec{I} = 3(\frac{13}{3}\vec{\imath}) - 3(-4\vec{\imath})$

=> $\vec{I} = 13\vec{\imath} + 12\vec{\imath}$

=> $\vec{I} = 25\vec{\imath}$ Ns

5 kg Sphere:

$\vec{I} = m_1\vec{v}_1 - m_1\vec{u}_1$

=> $\vec{I} = 5(1\vec{\imath}) - 5(6\vec{\imath})$

=> $\vec{I} = 5\vec{\imath} - 30\vec{\imath}$

=> $\vec{I} = -25\vec{\imath}$ Ns

Day 1: Classwork Questions: Pg 112/113 Qs 2/3/4/7

Day 2: Classwork Questions: Pg 112/113 Qs 8/9/13/15 and then try Q17 and Q18(i)(ii)

Topic 31: Oblique Collisions

- When two particles collide obliquely, it means that they collide while moving at an angle to one another.
- We handle oblique collisions by taking the line joining the centre of the particles, at the moment of impact, as the $\vec{\imath}$ axis.
- Because the particles are always smooth, the forces they exert on each other are always along the $\vec{\imath}$ axis, so the $\vec{\jmath}$ velocities remain unchanged by the impact.
- So, if two particles are moving at $a\vec{\imath} + b\vec{\jmath}$ and $c\vec{\imath} + d\vec{\jmath}$ before an impact, and $p\vec{\imath} + q\vec{\jmath}$ and $r\vec{\imath} + s\vec{\jmath}$ after the impact then there are 4 different equations we can use when two particles collide obliquely:

Eqn A	$\vec{\jmath}$ velocity of particle 1 stays the same	d = s
Eqn B	$\vec{\jmath}$ velocity of particle 2 stays the same	b = q
Eqn C	Law of Conservation of Momentum	$m_1a + m_2c = m_1p + m_2s$
Eqn D	Newton's Law of Restitution	$-e = \frac{p-q}{a-c}$

- **Example 1:** Pg 116 Ex 6C Q4

A sphere of mass 2m, moving with velocity $5\vec{\imath} + 5\vec{\jmath}$, collides obliquely with a sphere of mass m which is at rest. $\vec{\imath}$ is along their line of their centres at impact. If the coefficient of restitution is $\frac{1}{2}$, find:

i) the velocities of each sphere after impact

ii) the impulse imparted to each sphere during collision

iii) the percentage loss in kinetic energy

iv) the angle through which the heavier mass is deflected.

Solution:

i) First, we use equations A and B above to lay out the information we were given:

Before	Mass	After
$5\vec{\imath} + 5\vec{\jmath}$	2m	$p\vec{\imath} + 5\vec{\jmath}$
$0\vec{\imath} + 0\vec{\jmath}$	m	$q\vec{\imath} + 0\vec{\jmath}$

- And now we sub into equations C and D above:

Eqn C	Eqn D
$m_1 a + m_2 c = m_1 p + m_2 s$ => $(2m)(5) + (m)(0) = (2m)(p) + (m)(q)$ => $10m = 2mp + mq$ => $2p + q = 10$……..Eqn 1	$-e = \frac{p-q}{a-c}$ => $-e = \frac{p-q}{5-0} = -\frac{1}{2}$ => $\frac{p-q}{5} = -\frac{1}{2}$ => $2p - 2q = -5$…….Eqn 2

We can now solve equations 1 and 2 together to find p and q:

Eqn 1: 2p + q = 10

Eqn 2 x -1: -2p + 2q = 5

=> 3q = 15

=> q = 5

And subbing into equation 1 gives:

Eqn 1: 2p + (5) = 10

2p = 10 - 5

2p = 5

p = $\frac{5}{2}$

- So, the velocities after impact will be $\frac{5}{2}\vec{\imath} + 5\vec{\jmath}$ and $5\vec{\imath} + 0\vec{\jmath}$.

ii) The impulses imparted to the two spheres are:

Sphere 1	Sphere 2
$\vec{I_1} = m_1\vec{v}_1 - m_1\vec{u}_1$ => $\vec{I_1} = 2m(\frac{5}{2}\vec{\imath} + 5\vec{\jmath}) - 2m(5\vec{\imath} + 5\vec{\jmath})$ => $\vec{I_1} = 5m\vec{\imath} + 10m\vec{\jmath} - 10m\vec{\imath} - 10m\vec{\jmath}$ => $\vec{I_1} = -5m\vec{\imath}$ Ns	$\vec{I_2} = m_2\vec{v}_2 - m_2\vec{u}_2$ => $\vec{I_2} = m(5\vec{\imath} + 0\vec{\jmath}) - m(0\vec{\imath} + 0\vec{\jmath})$ => $\vec{I_2} = 5m\vec{\imath}$ => $\vec{I_2} = 5m\vec{\imath}$ Ns

iii) As we are asked for "percentage energy loss", we can use our shortcut of using $\vec{\imath}$ components only to calculate the loss, but we will have to use $\vec{\imath}$ and $\vec{\jmath}$ components to calculate the total energy in the system before the collision:

- Before we work out the kinetic energies we need to work out the magnitudes of all 4 velocities:

$$|5\vec{\imath} + 5\vec{\jmath}| = \sqrt{(5)^2 + (5)^2} = \sqrt{50}$$

$$\left|\frac{5}{2}\vec{\imath} + 5\vec{\jmath}\right| = \sqrt{(\frac{5}{2})^2 + (5)^2} = \sqrt{\frac{125}{4}}$$

$$|5\vec{\imath} + 0\vec{\jmath}| = \sqrt{(5)^2 + (0)^2} = 5$$

- We can now work out the kinetic energies before and after the collision:

Before	After
$KE_{before} = \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2$ $KE_{before} = \frac{1}{2}(2m)(\sqrt{50})^2 + \frac{1}{2}(m)(0)^2$ => $KE_{before} = (m)(50) + 0$ => $KE_{before} = 50m$	$KE_{after} = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$ => $KE_{after} = \frac{1}{2}(2m)\left(\sqrt{\frac{125}{4}}\right)^2 + \frac{1}{2}(m)(5)^2$ => $KE_{after} = \frac{125m}{4} + \frac{25m}{2}$ => $KE_{after} = \frac{125m}{4} + \frac{50m}{4} = \frac{175m}{4}$

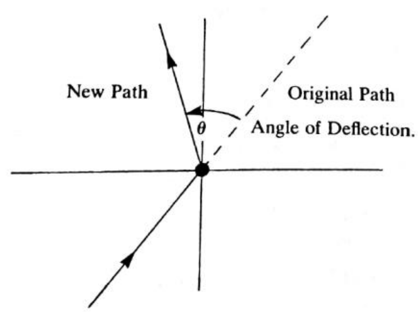
- So, the loss in Kinetic Energy is:

$$= 50m - \frac{175m}{4}$$
$$= \frac{200m}{4} - \frac{175m}{4}$$
$$= \frac{25m}{4}$$

- So, the percentage energy lost will be:

$$= \frac{\frac{25m}{4}}{50m} \times \frac{100}{1}$$
$$= \frac{1}{8} \times \frac{100}{1}$$
$$= 12.5\%$$

iv) A diagram of the situation is:



- The slope of the original path = $\frac{\vec{j}\,component}{\vec{i}\,component} = \frac{5}{5} = 1$ since $\vec{u} = 5\vec{i} + 5\vec{j}$.
- The slope of the new path = $\frac{\vec{j}\,component}{\vec{i}\,component} = \frac{5}{\frac{5}{2}} = 2$ since $\vec{v} = 5\vec{i} + \frac{5}{2}\vec{j}$.
- The angle between the two lines is given by a formula you have come across on your Maths course:

$$\tan\theta = \pm\frac{m_1 - m_2}{1 + m_1m_2}$$
=> $\tan\theta = \pm\frac{1-2}{1+(1)(2)}$
=> $\tan\theta = \pm\frac{-1}{3}$ => $\tan\theta = \pm\frac{1}{3}$

- The acute angle is given by the positive value, so:

$\tan\theta = \frac{1}{3}$ => $\theta = \tan^{-1}(\frac{1}{3})$ => $\theta = 18.43°$.

Day 1: Classwork Questions: Pg 116 Ex 7C Qs 1 – 3

Day 2: Classwork Questions: Pg 116 Ex 7C Qs 5/7 - 9

- Let's look at more complicated problems.

Note:

- Two types of problem that have arisen in the past can be spotted by watching for particular key phrases as shown below.
- If you spot these phrases, then the initial set up of the "Before Mass After" table is slightly different.

If "Angle of particle after impact" is given or mentioned, the setup is:

Before	Mass	After
$u\cos A\,\vec{i}+u\sin A\,\vec{j}$	m	$v\cos B\,\vec{i}+v\sin B\,\vec{j}$
$0\vec{i}+0\vec{j}$	m	$p\vec{i}+0\vec{j}$

If "angle of deflection" or "deflected through a certain angle" is mentioned then the setup is:

Before	Mass	After
$u\cos A\,\vec{i}+u\sin A\,\vec{j}$	m	$p\vec{i}+u\sin A\,\vec{j}$
$0\vec{i}+0\vec{j}$	m	$q\vec{i}+0\vec{j}$

- Example 2: Extra Sheet Set A Q7

A smooth sphere A moving with speed $5u$, collides with an identical smooth sphere B which is at rest. The direction of motion of A, before impact, makes an angle of $\tan^{-1}\frac{3}{4}$ with the line of centres at the instant of impact. The coefficient of restitution between the spheres is e. Show that the direction of motion of A is deflected through an angle θ where $\tan\theta=\dfrac{6(1+e)}{17-8e}$

Solution:

- As "deflected through" is mentioned, we will setup as described above:

Before	Mass	After
$ucosA\,\vec{i}+usinA\,\vec{j}$	m	$p\vec{i}+usinA\vec{j}$
$0\vec{i}+0\vec{j}$	m	$q\vec{i}+0\vec{j}$

- Asas
- As $\tan A=\frac{3}{4}$ => $\sin A=\frac{3}{5}$ and $\cos A=\frac{4}{5}$, so the velocity of the sphere before impact will be:

$$=5u\cos A\,\vec{i}+5u\sin A\,\vec{j}$$
$$=5u\left(\tfrac{4}{5}\right)\vec{i}+5u\left(\tfrac{3}{5}\right)\vec{j}$$
$$=4u\vec{i}+3u\vec{j}$$

- So our table will be:

Before	Mass	After
$4u\,\vec{i}+3u\,\vec{j}$	m	$p\vec{i}+3u\vec{j}$
$0\vec{i}+0\vec{j}$	m	$q\vec{i}+0\vec{j}$

- And now we sub into equations C and D as before:

Eqn C	Eqn D
$m_1a + m_2c = m_1p + m_2s$ => $(m)(4u) + (m)(0) = (m)(p) + (m)(q)$ => $4um = mp + mq$ => $p + q = 4u$………Eqn 1	$-e = \frac{p-q}{4u-0}$ => $-e = \frac{p-q}{4u}$ => $p - q = -4eu$………Eqn 2

We can now solve equations 1 and 2 together to find p and q:

Eqn 1: p + q = 4u
Eqn 2: <u>p - q = -4eu</u>
=> 2p = 4u - 4eu
=> p = 2u - 2eu

And subbing into equation 1 gives:

Eqn 1: p + q = 4u
2u - 2eu + q = 4u
=> q = 2u + 2eu

> Mention that it can be easier to just re-solve equations 1 and 2 and eliminate q instead.
>
> Also mention that if it's a angle deflected by A, then we don't need to find q at all.

- As we did in Example 1, we now form expressions for the slopes of A's path before and after the collision:
- The slope of the original path = $\frac{\vec{j}\,component}{\vec{i}\,component} = \frac{3u}{4u} = \frac{3}{4}$ since $\vec{u} = 3u\vec{i} + 4u\vec{j}$.
- The slope of the new path = $\frac{\vec{j}\,component}{\vec{i}\,component} = \frac{3u}{2u-2eu} = \frac{3}{2-2e}$ since $\vec{v} = (2u - 2eu)\vec{i} + 3u\vec{j}$.
- And using the angle between two lines formula again:

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$$

$$=> \tan\theta = \left|\frac{\frac{3}{4} - \frac{3}{2-2e}}{1 + (\frac{3}{4})(\frac{3}{2-2e})}\right|$$

$$=> \tan\theta = \left|\frac{\frac{6-6e-12}{8-8e}}{1 + \frac{9}{8-8e}}\right|$$ (tidying up top and bottom into single fractions)

$$=> \tan\theta = \left|\frac{\frac{-6-6e}{8-8e}}{\frac{8-8e}{8-8e} + \frac{9}{8-8e}}\right|$$ (changing 1 to $\frac{8-8e}{8-8e}$)

$$=> \tan\theta = \left|\frac{\frac{-6-6e}{8-8e}}{\frac{17-8e}{8-8e}}\right|$$ (adding together two fractions in denominator)

$$=> \tan\theta = \left|\frac{-6-6e}{17-8e}\right|$$ (multiplying above and below by 8 - 8e)

$$=> \tan\theta = \frac{6+6e}{17-8e}$$ (applying the modulus to take the positive value)

$$=> \tan\theta = \frac{6(1+e)}{17-8e}$$ (factorising out the 6 on top)

Q.E.D.

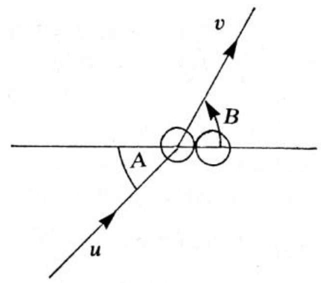
Classwork Questions: Extra Sheet Set A Qs 9/19

- Example 3: Pg 117 Ex 6D Q1

  A smooth sphere collides obliquely with an equal sphere, which is at rest. Before the impact, the line of motion of the first sphere makes an angle A with their line of centres; afterwards this angle is B. The coefficient of restitution, is $\frac{1}{4}$, show that $8\tan A = 3\tan B$.

  Solution:

  - A diagram of the situation is shown below.
  - We will let ‘u’ be the speed before the collision and ‘v’ be the speed after the collision.



  - Note that in yesterday’s questions, there was no mention of this new angle ‘B’ and only the angle that A is “deflected through” was mentioned.
  - As a new angle B is mentioned, our starting setup will be:

Before	Mass	After
$u\cos A\,\vec{i} + u\sin A\,\vec{j}$	m	$v\cos B\,\vec{i} + v\sin B\,\vec{j}$
$0\vec{i} + 0\vec{j}$	m	$p\vec{i} + 0\vec{j}$

  - Now, using equations A and B we can equate the $\vec{j}$ components for mass m to get:

    $u\sin A = v\sin B$……….Eqn 1
  - And now we sub into equations C and D:

Eqn C	Eqn D
$m_1a + m_2c = m_1p + m_2s$ => $(m)(u\cos A) + (m)(0) = (m)(v\cos B) + (m)(p)$ => $mu\cos A = mv\cos B + mp$ => $u\cos A = v\cos B + p$ => $p = u\cos A - v\cos B$………Eqn 2	=> $-e = \frac{v\cos B - p}{u\cos A - 0} = -\frac{1}{4}$ => $\frac{v\cos B - p}{u\cos A} = -\frac{1}{4}$ => $4(v\cos B - p) = -1(u\cos A)$ => $4v\cos B - 4p = -1u\cos A$ => $4p = 4v\cos B + u\cos A$ => $p = v\cos B + \frac{1}{4}u\cos A$……Eqn 3

We now have two expressions for p so we can equate our two equations above:

$$u\cos A - v\cos B = v\cos B + \frac{1}{4}u\cos A$$

=> $4u\cos A - 4v\cos B = 4v\cos B + u\cos A$

=> $4u\cos A - u\cos A = 4v\cos B + 4v\cos B$

=> $3u\cos A = 8v\cos B$……..Eqn 4

Mention that we can also just solve Eqns 2 and 3 together as before and eliminate p.

- Finally, we divide equations 1 and 4 together:

=> $\frac{u \sin A}{3u \cos A} = \frac{v \sin B}{8v \cos B}$ (dividing equations 1 and 4)

=> $\frac{\sin A}{3 \cos A} = \frac{\sin B}{8 \cos B}$ (cancelling the u's and v's)

=> $\frac{\tan A}{3} = \frac{\tan B}{8}$ (as $\tan A = \frac{\sin A}{\cos A}$ and $\tan B = \frac{\sin B}{\cos B}$)

=> $8 \tan A = 3 \tan B$ Q.E.D.

Day 1: Classwork Questions: Extra Sheet Set A Q8 (1st proof) and then Pg 117 Ex 6D Qs 2/3
Day 2: Classwork Questions: Pg 117 Ex 6D Qs 4/5

- Topic 32: Bouncing Projectiles
  - The key to handling problems involving bouncing particles is to remember something we discovered in an earlier topic (Topic 25 Example 2):
  - If a particle is moving at velocity $a\vec{\imath} - b\vec{\jmath}$ and it collides with a horizontal surface, then the collision only takes place in the $\vec{\jmath}$ direction and the velocity in the $\vec{\imath}$ direction remains the same
    - This means the new velocity after collision/bounce will be: $a\vec{\imath} + eb\vec{\jmath}$.
  - Similarly, if a particle is moving at velocity $p\vec{\imath} + q\vec{\jmath}$ and it collides with a vertical surface, then the collision only takes place in the $\vec{\imath}$ direction and the velocity in the $\vec{\jmath}$ direction remains the same
    - This means the new velocity after collision/bounce will be: $-ep\vec{\imath} + q\vec{\jmath}$.
  - To summarise:

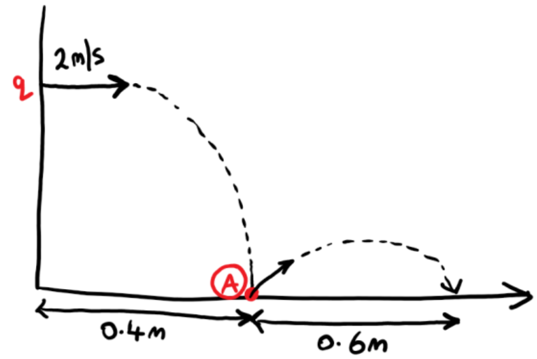
Collision/Bounce off Horizontal Surface		Collision/Bounce off Vertical Surface	
Velocity Before $a\vec{\imath} - b\vec{\jmath}$	Velocity After $a\vec{\imath} + eb\vec{\jmath}$	Velocity Before $p\vec{\imath} + q\vec{\jmath}$	Velocity After $-ep\vec{\imath} + q\vec{\jmath}$

- Example: Pg 119 Ex 6E Q3

A ball is projected horizontally from a point q above a smooth horizontal plane with speed 2 m/s. The ball first hits the plane at a point whose horizontal displacement from q is 0.4 m. The ball next strikes the plane at a horizontal displacement of 1 m from q. The coefficient of restitution between the ball and the plane is e. Find the value of e.

Solution:

- Let's start with a diagram of the situation:



- The initial velocity of the particle is $2\vec{\imath} + 0\vec{\jmath}$ as it's projected horizontally.
- We can now form our 4 equations of motion:

$v_x = 2$	$v_y = 0 - gt$ => $v_y = -gt$
$s_x = 2t$	$s_y = 0t - \frac{1}{2}gt^2$ => $s_y = -\frac{1}{2}gt^2$

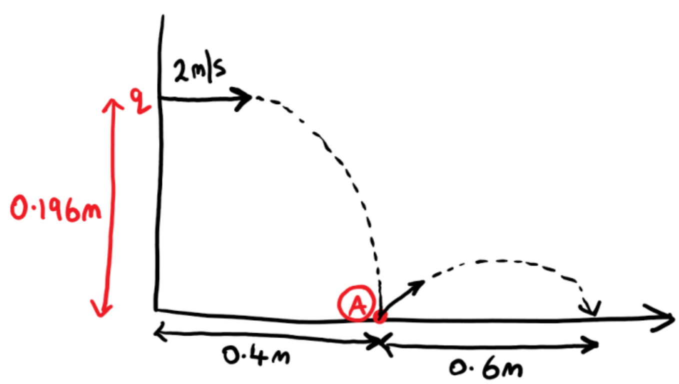
- First, we will work out the velocity of the particle right before the first bounce i.e. v_x and v_y when S_x = 0.4.

$s_x = 2t = 0.4$
=> $t = 0.2$ secs

- It will be useful to know how far below the ground is from q later on in the question, so we will also work out S_y at this time:

$$s_y = -\frac{1}{2}gt^2 = -\frac{1}{2}(9.8)(0.2)^2 = -0.196 \text{ m}$$

- So, our updated diagram will be:



- We already know v_x is a fixed value of 2 all the time, but $v_y = -gt = -(9.8)(0.2) = -1.96$.
- So, the velocity right before the bounce will be: $2\vec{\imath} - 1.96\vec{\jmath}$.
- Using our result from above, the velocity right after the bounce so will be: $2\vec{\imath} + e1.96\vec{\jmath}$

- We are now leaving point A in the diagram above and we can effectively treat it as a new projectiles problem, so we form 4 new equations with initial velocity being: $2\vec{i} + 1.96e\vec{j}$

$v_x = 2$	$v_y = 1.96e - gt$
$s_x = 2t$	$s_y = 1.96et - \frac{1}{2}gt^2$

- We now use the fact that we know $s_y = -0.196$ the next time the projectile bounces on the plane i.e. when $s_x = 0.6$:

$s_x = 2t = 0.6$

=> $t = 0.3$ secs

- So:

$$s_y = 1.96et - \frac{1}{2}gt^2 = -0.196$$

=> $1.96e(0.3) - \frac{1}{2}(9.8)(0.3)^2 = -0.196$

=> $0.588e - 0.441 = -0.196$

=> $0.588e = -0.196 + 0.441$

=> $0.588e = 0.245$

=> $e = \frac{0.245}{0.588} = \frac{5}{12}$

Need to check again as answer isn't right.

Classwork Questions: Pg 119/120 Ex 6E Qs 1/2/7 and then try Qs 4/6

Revision Questions and Test