

➤ Chapter 8: Difference Equations

➤ Topic 37: Recap on JC Patterns

✚ Went through all this page on the board getting them to work with me at the back of their copies.

• Linear Patterns from JC:

- A **linear** sequence of numbers is a list of numbers where there is a **common difference** between each term.
 - E.g. 3, 8, 13, 18, 23.....which has a common difference of +5.
- In Junior Cycle, we learned how to find the **General Term** (T_n) of a linear sequence.

Step 1: Multiply the common difference by 'n'.....in the sequence above: $5n$
Step 2: See what needs to be added or subtracted to $5n$ to get each of the terms in the sequence.....in the sequence above: -2
Step 3: Write the General Term $T_n = 5n - 2$.

- Once we had the general term, there were two useful things that we were able to do:

Finding any term in the sequence	Finding the term number of a value
For example, to find the 50 th term: $T_n = 5n - 2$ $\Rightarrow T_n = 5(50) - 2$ $= 250 - 2$ $= \mathbf{248}$	For example, what term is 568? $5n - 2 = 568$ $\Rightarrow 5n = 570$ $\Rightarrow n = \mathbf{114}$ $\Rightarrow \text{the } 114^{\text{th}} \text{ term is } 568$

• Quadratic Patterns from JC:

- We also learned in Junior Cycle about **Quadratic Sequences**.
- A quadratic sequence is a list of numbers where the **second difference** between each term is the same every time.
 - E.g. 4, 7, 12, 19, 28, 39.....which has **first differences** of +3, +5, +7, +9.....and a **second difference** of +2.
- To find the General Term of a quadratic sequence we:

Step 1: Let the General Term $T_n = an^2 + bn + c$.
Step 2: The second difference represents $2a$, so halving the second difference gave us a value for ain the sequence above, the second difference is +2, so 'a' would be 1.
Step 3: Use two of the terms in the sequence to make two simultaneous equations, which we solve to find 'b' and 'c'.....

$T_n = an^2 + bn + c$ $T_2 = (2)^2 + b(2) + c = 7$ $\Rightarrow 4 + 2b + c = 7$ $\Rightarrow 2b + c = 3 \dots \text{Eqn 1}$	$T_3 = (3)^2 + b(3) + c = 12$ $\Rightarrow 9 + 3b + c = 12$ $\Rightarrow 3b + c = 3 \dots \text{Eqn 2}$
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Solving Equations 1 and 2 gives $b = 0$ and $c = 3$
 $\Rightarrow T_n = n^2 + (0)n + 3$
 $\Rightarrow T_n = \mathbf{n^2 + 3}$

- **Note:** An alternative method to find T_n of a Quadratic Sequence is use three terms and form three equations in a , b and c and solve for those values then.

- If a sequence is such that the **third difference** between its terms **is the same** every time, then that is known as a **cubic sequence**.

E.g. 11, 31, 69, 131, 223.....

➤ Topic 38: Arithmetic Sequences/Series

- In your LC Maths course, the term we use to describe linear sequences is **Arithmetic Sequences**.
- We use a formula at senior cycle to help us find the *General Term* of this type of sequence quickly:

$$T_n = a + (n - 1)d$$

See pg22
Tables Book

where 'a' is the 1st term and 'd' is the common difference

- If we add together the terms of an arithmetic sequence, we get an **Arithmetic Series**.
- It can be useful to be able to find the sum of the terms of an arithmetic series.
- We use a formula to help us find the sum of the first n terms:

$$S_n = \frac{n}{2} \{2a + (n - 1)d\}$$

See pg22
Tables Book

where 'a' is the 1st term and 'd' is the common difference

- **Example:** A sequence is 3, 9, 15, 21, 27.....

- Find the 60th term.
- Find the sum of the first 60 terms.
- What term is the first term to be bigger than 10000?

Solution:

i) Firstly, we will find the *General Term*:

- As $a = 3$ and $d = 6$, then

$$T_n = a + (n - 1)d$$

$$\Rightarrow T_n = 3 + (n - 1)(6)$$

$$\Rightarrow T_n = 3 + 6n - 6$$

$$\Rightarrow T_n = 6n - 3$$

- So, now we can find the 60th term:

$$T_{60} = 6(60) - 3$$

$$= 360 - 3$$

$$= \mathbf{357}$$

ii) This time we need to use the S_n formula:

$$S_n = \frac{n}{2} \{2a + (n - 1)d\}$$

$$= \frac{60}{2} \{2(1) + (60 - 1)(3)\}$$

$$= 30 \{2 + 177\}$$

$$= \mathbf{5370}$$

iii) To find this term, we need to solve $T_n > 10000$

$$6n - 3 > 10000$$

$$\Rightarrow 6n > 10003$$

$$\Rightarrow n > 1667.17$$

$$\Rightarrow \text{the } \mathbf{1668^{\text{th}}}$$
 term will be the first term to be bigger than 10000

Day 1: Classwork Questions: Sheet: Exercise 1 Qs 4/5/9 and Exercise 2 Qs 1(ii)(vi)/4 and then give Pg 146 Ex 8A Qs 2/3/5 for HW to finish also

➤ Topic 39: Geometric Sequences:

- A **Geometric sequence** is a set of numbers where each term is found by multiplying the previous term by the same number, known as the **common ratio**.
E.g. 10, 30, 90, 270.....
- The common ratio is denoted 'r'.
- We also use a formula to help us find the *General Term* of this type of sequence:

$$T_n = ar^{n-1}$$

See pg22
Tables Book

where 'a' is the 1st term and 'r' is the common ratio.

- Similarly, a **Geometric Series** is a series where the terms of a Geometric Sequence are added together.
- It can be easily shown that the sum of the first n terms of a Geometric Series can be found using the formula:

$$S_n = \frac{a(1-r^n)}{1-r} \text{ if } r < 1 \text{ or } S_n = \frac{a(r^n-1)}{r-1} \text{ if } r > 1$$

See pg22
Tables Book

where 'a' is the 1st term and 'r' is the common ratio.

- In the specific case of an infinite Geometric series, the formulae above simplify to:

$$S_\infty = \frac{a}{1-r}$$

Day 1: Questions from Sheet on Geo 5 Ex 4 Qs 1(ii)(iv)(v)(vi)/2(ii)(iv)/3 and Ex 5 Qs 1(ii)/2(ii)
Day 2: Classwork Questions: Pg 146 Ex 8A Qs 8/10/11/13

➤ Topic 40: Recurrence Relations

- A **recurrence relation** is a sequence which is defined differently to how previous sequences we have come across are defined.
- Up to now, the *General Term* described any term in terms of 'n'.
- In a **recurrence relation** a sequence is defined showing how any term is connected to the previous term.
- Examples of recurrence relations would be:

Note the use of 'u' instead of 'T'

i) $T_n = 3T_{n-1}$
Any term = 3 times the previous term

ii) $T_{n+1} = T_n - 4$
Any term = the previous term less 4

iii) $u_{n+1} = \frac{1}{2}u_n - 3$
Any term = half the previous term less 3

- **Example:** Pg 149 Ex 8B Q5

A sequence $T_1, T_2, T_3, T_4, T_5, T_6, \dots$ is defined by the recurrence relation $T_{n+1} = \frac{1}{4}T_n$, for $n \in \mathbb{N}$ where $T_1 = 144$.

- Write down the first four terms.
- Find S_∞ the sum of the entire series.

Solution:

- Firstly, the key information we need is the recurrence relation itself $T_{n+1} = \frac{1}{4}T_n$, which tells us that any term in this sequence is a quarter of the previous term.

- We were also given $T_1 = 144$, so the next term will be $\frac{1}{4}(144) = 36$.
- The third term will be $\frac{1}{4}(36) = 9$ and the fourth term will be $\frac{1}{4}(9) = \frac{9}{4}$

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- Let's start by writing out the sequence and identifying what type it is:

$$144, 36, 9, \frac{9}{4}, \dots$$

- This is a Geometric Sequence with $a = 144$ and $r = \frac{1}{4}$.
- To find S_∞ , we use the formula above for the sum of an infinite Geometric sequence:

$$S_\infty = \frac{a}{1-r}$$

$$\Rightarrow S_\infty = \frac{144}{1-\frac{1}{4}} = \frac{144}{\frac{3}{4}} = 192$$

Classwork Questions: Pg 149 Ex 8B Qs 2/3/7/8/10

➤ Topic 41: First Order Difference Equations

- We saw in the previous topic that when we are given a recurrence relation, we have to go through the process of working out the previous term if we want to know ANY term in the sequence.
- This can be a fairly time-consuming process, if for example, we wanted to know the 5000th term of a sequence, as we'd have to work out the previous 4999 terms first.
- For this reason, it is very useful if we can work out the **General Term** from a recurrence relation.
- In order to do this, we have to learn how to solve a **difference equation**.

• First Order Difference Equations:

- These difference equations are ones where each term is defined in terms of one previous term. e.g. $T_{n+1} = \frac{1}{4}T_n - 3$

• **Example 1:** Pg 151 Ex 8C Q3

- i) Solve the difference equation $u_n = 3 + 2u_{n-1}$ given that $u_0 = 1$.
- ii) As $n \rightarrow \infty$, which of these is true? A: u_n gets smaller and smaller. B: u_n gets bigger and bigger. C: u_n tends to a finite limit k.

Solution:

- The strategy to solve this type is to sub in increasing values for n starting at 1, and see if we can spot a pattern to link u_n and the term we've been given (in this example u_0):

<u>n = 1</u>	<u>n = 2</u>	<u>n = 3</u>
$u_n = 3 + 2u_{n-1}$	$u_n = 3 + 2u_{n-1}$	$u_n = 3 + 2u_{n-1}$
$\Rightarrow u_1 = 3 + 2u_0$	$\Rightarrow u_2 = 3 + 2u_1$	$\Rightarrow u_3 = 3 + 2u_2$
$\Rightarrow u_1 = 3 + 2(1)$	$\Rightarrow u_2 = 3 + 2(3 + 2(1))$	$\Rightarrow u_3 = 3 + 2(3 + 2(3) + 2^2(1))$
	$\Rightarrow u_2 = 3 + 2(3) + 2^2(1)$	$\Rightarrow u_3 = 3 + 2(3) + 2^2(3) + 2^3(1)$

- Hopefully, you can now spot from looking at u_1, u_2 and u_3 above that
 - the terms in red form a pattern and are of the form $2^n(1)$
 - the terms in blue form a geometric series with $a = 3$ and $r = 2$, so we can find the sum of that series using our formula from the first topic:

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$\Rightarrow S_n = \frac{3((2)^n - 1)}{2 - 1}$$

$$\Rightarrow S_n = \frac{3((2)^n - 1)}{1}$$

$$\Rightarrow S_n = 3(2)^n - 3$$

- So, combining the two gives us: $1(2)^n + 3(2)^n - 3$.
- And finally, adding together the like terms gives us our final solution: $4(2)^n - 3$

ii)

- As we now know the General Term, we can write out the first few terms of the sequence:

$$u_1 = 4(2^1) - 3 = 5, u_2 = 4(2^2) - 3 = 13, u_3 = 4(2^3) - 3 = 29, u_4 = 4(2^4) - 3 = 61$$

$$\Rightarrow \text{Sequence is: } 5, 13, 29, 61, \dots$$
- We can see from the sequence above that the distance between the terms is increasing as we add on more terms, so u_n is just going to get bigger and bigger as $n \rightarrow \infty$.

• **Example 2:** Pg 151 Ex 8C Q4

Savani has €500 in her savings account. She decides that, now that she has a new job, she will put €1000 on January 1st every year into her savings account. The bank offers 1% compound interest per annum.

i) Show that the amount in her savings account after n years (A_n) is determined by the difference equation $A_n = 1.01A_{n-1} + 1000$.

ii) Given that $A_0 = 500$, solve this difference equation.

iii) How much will she have in her account in 20 years' time?

Solution :

i) If she has 'x' euro in her account at the start of ANY year, then she will have 1.01x at the end of the year as the bank adds 1% compound interest.

- She then adds in €1000 at the start of the next year.

- So, at the end of ANY year she will have 1.01 of what she had the previous year plus an additional €1000 $\Rightarrow A_n = 1.01A_{n-1} + 1000$ Q.E.D

ii) As before,

$n = 1$	$n = 2$	$n = 3$
$A_n = 1.01A_{n-1} + 1000$	$A_n = 1.01A_{n-1} + 1000$	$A_n = 1.01A_{n-1} + 1000$
$\Rightarrow A_1 = 1.01A_0 + 1000$	$\Rightarrow A_2 = 1.01A_1 + 1000$	$\Rightarrow A_3 = 1.01A_2 + 1000$
$A_1 = 1.01(500) + 1000$	$\Rightarrow A_2 = 1.01(1.01(500) + 1000) + 1000$	$= 1.01((1.01^2)(500) + 1.01(1000) + 1000) + 1000$
	$A_2 = (1.01^2)(500) + 1.01(1000) + 1000$	$= (1.01^3)(500) + (1.01^2)1000 + (1.01)1000 + 1000$

- So hopefully, we see the pattern again.....

- The red terms being $(1.01^n)(500)$ and the blue terms being a Geometric Series with $a = 1000$ and $r = 1.01$.

- We can find the sum of this series using our formula:

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$\Rightarrow S_n = \frac{1000((1.01)^n - 1)}{1.01 - 1}$$

$$\Rightarrow S_n = \frac{1000((1.01)^n - 1)}{0.01}$$

$$\Rightarrow S_n = 100,000((1.01)^n - 1)$$

- So, our solution will be: $(1.01^n)(500) + 100,000((1.01)^n - 1)$.

$$= 500(1.01^n) + 100,000(1.01)^n - 100,000$$

$$\Rightarrow A_n = 100,500(1.01)^n - 100,000$$

iii) To calculate how much she will have after 20 years, we just have to sub in $n = 20$:

$$A_n = 100,500(1.01)^n - 100,000$$

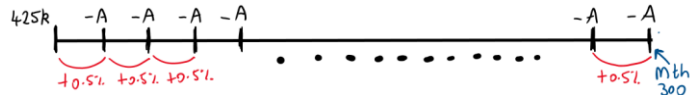
$$A_{20} = 100,500(1.01)^{20} - 100,000$$

$$\Rightarrow A_{20} = \text{€}22,629.10$$

Day 1: Classwork Questions: Pg 151 Ex 8C Qs 1/2/5(a)(b)

Day 2: Classwork Questions: Pg 152 Ex 8C Qs 6/8/9/11

• Interest Repayments:



• **Example:** Pg 154 Ex 8D Q3

A couple borrows €425,000 to buy a house. They will repay the same amount (A) each month for 25 years. The Building Society charges a monthly interest rate of 0.5%.

- i) How many monthly repayments will there be ?
- ii) If D_n is the amount of debt owing after n months write down a difference equation in D_n .
- iii) Solve the difference equation.
- iv) Find A to the nearest euro.

Solution:

- i) The number of monthly repayments over 25 years will be $25 \times 12 = 300$.
- ii) In ANY month, the couple will owe 1.005 of what they owe the previous month D_{n-1} (as the interest rate is 0.5%) and then they will make a repayment of A
 $\Rightarrow D_n$ will be: $1.005(D_{n-1}) - A$

iii) We now solve as before:

<u>$n = 1$</u>	<u>$n = 2$</u>	<u>$n = 3$</u>
$D_n = 1.005D_{n-1} - A$	$D_n = 1.005D_{n-1} - A$	$D_n = 1.005D_{n-1} - A$
$\Rightarrow D_1 = 1.005D_0 - A$	$\Rightarrow D_2 = 1.005D_1 - A$	$\Rightarrow D_3 = 1.005D_2 - A$
	$\Rightarrow D_2 = 1.005(1.005D_0 - A) - A$	$\Rightarrow D_3 = 1.005(1.005^2D_0 - 1.005A - A) - A$
	$\Rightarrow D_2 = 1.005^2D_0 - 1.005A - A$	$\Rightarrow D_3 = 1.005^3D_0 - 1.005^2A - 1.005A - A$
		$\Rightarrow D_3 = 1.005^3D_0 - 1(1.005^2A + 1.005A + A)$

- So the pattern this time is.....
 - The first term being $(1.005^n)D_0$ and the terms in the brackets being a Geometric Series with $a = A$ and $r = 1.005$.
- Again, we can find the sum of this series using our formula:

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$\Rightarrow S_n = \frac{A((1.005)^n - 1)}{1.005 - 1}$$

$$\Rightarrow S_n = \frac{A((1.005)^n - 1)}{0.005}$$

$$\Rightarrow S_n = 200A((1.005)^n - 1)$$

- So, our solution will be: $D_n = (1.005^n)D_0 - 200A((1.005)^n - 1)$.

iv)

- We know $D_0 = €425,000$ and $n = 300$ from part (i) and we also know that the debt will be repaid when $D_n = 0 \Rightarrow$

$$\begin{aligned} \Rightarrow (1.005^{300})(425,000) - 200A((1.005)^{300} - 1) &= 0 \\ \Rightarrow 1897612.17 - 200A(3.464969812) &= 0 \\ \Rightarrow 1897612.17 - 692.9939624A &= 0 \\ \Rightarrow 1897612.17 &= 692.9939624A \\ \Rightarrow A = \frac{1897612.17}{692.9939624} &= €2738.28 = \mathbf{€2738} \end{aligned}$$

Put this in memory M on Calculator

Classwork Questions: Pg Ex 8D Qs 1/2/5

➤ Topic 42: Second Order Difference Equations

• Second Order Homogeneous Equations:

- These difference equations are ones where each term is defined in terms of two previous terms. e.g. $T_{n+1} = 3T_n - \frac{3}{4}T_{n-1}$ or $u_{n+2} - 2u_{n+1} + 3u_n = 0$
- Second order difference equations are called **homogeneous**, if they only contain T_n, T_{n+k} or T_{n-k} .
- If they contain other terms other than those, they are known as **inhomogeneous** equations or **non-homogeneous** ones. E.g. $u_{n+2} - 2u_{n+1} + 3u_n - 4 = 0$
- The example above is non-homogeneous as it contains an extra constant of 4.
- When solving homogeneous equations, we first solve what is known as the **characteristic** quadratic equation.
- If the difference equation is: $u_{n+2} - 2u_{n+1} + 3u_n = 0$ then the corresponding characteristic equation will be: $x^2 - 2x + 3 = 0$. (Note that the coefficients have to match in both equations).
- There are two theorems we need then to solve the difference equation:

Difference Equation Theorem 1	Difference Equation Theorem 2
If a and b are the two roots of the characteristic equation $px^2 + qx + r = 0$, then the solution to the second-order difference equation $pu_{n+2} + qu_{n+1} + ru_n = 0$ will be in the form: <div style="text-align: center; border: 2px solid red; padding: 5px; margin-top: 10px;"> $u_n = la^n + mb^n$ </div>	If a and a are the two equal roots of the characteristic equation $px^2 + qx + r = 0$, then the solution to the second-order difference equation $pu_{n+2} + qu_{n+1} + ru_n = 0$ will be in the form: <div style="text-align: center; border: 2px solid red; padding: 5px; margin-top: 10px;"> $u_n = la^n + mna^n$ </div>

- **Example 1:** Pg 156 Ex 8E Q4 Solve $2u_{n+2} - 11u_{n+1} + 5u_n = 0$ given $u_0 = 2$ and $u_1 = -8$.

Solution:

- First we form the characteristic equation and solve it:

$2u_{n+2} - 11u_{n+1} + 5u_n = 0$ will have characteristic equation

$$2x^2 - 11x + 5 = 0$$

$$\Rightarrow (2x - 1)(x - 5) = 0$$

$$\Rightarrow x = \frac{1}{2} \quad \text{or} \quad x = 5$$

- Using Theorem 1 above with our two roots $a = \frac{1}{2}$ and $b = 5$, our solution will be in the form:

$$u_n = la^n + mb^n$$

$$\Rightarrow u_n = l\left(\frac{1}{2}\right)^n + m(5)^n$$

- Now we use the other information we were given in the question:

<p>If $u_0 = 2$:</p> $\Rightarrow u_0 = l\left(\frac{1}{2}\right)^0 + m(5)^0 = 2$ $\Rightarrow l + m = 2 \quad (\text{as } \left(\frac{1}{2}\right)^0 \text{ and } (5)^0 = 1)$	<p>If $u_1 = -8$:</p> $\Rightarrow u_1 = l\left(\frac{1}{2}\right)^1 + m(5)^1 = -8$ $\Rightarrow \frac{1}{2}l + 5m = -8$ $\Rightarrow l + 10m = -16 \quad (\text{multiplying by 2})$
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- We now solve the two simultaneous equations above giving $l = 4$ and $m = -2$.

- So, the solution to our difference equation is: $u_n = 4\left(\frac{1}{2}\right)^n - 2(5)^n$.

- **Example 2:** Pg 156 Ex 8E Q9 Solve $u_{n+1} - 4u_n + 4u_{n-1} = 0$ given $u_0 = -1$ and $u_1 = 8$.

Solution:

- Again, we start by forming the characteristic equation and solving it:

$u_{n+1} - 4u_n + 4u_{n-1} = 0$ will have characteristic equation

$$x^2 - 4x + 4 = 0$$

$$\Rightarrow (x - 2)(x - 2) = 0$$

$$\Rightarrow x = 2 \quad \text{or} \quad x = 2$$

- Using Theorem 2 above with our two roots $a = 2$ and $a = 2$, our solution will be in the form:

$$u_n = la^n + mna^n$$

$$\Rightarrow u_n = l(2)^n + mn(2)^n$$

- Using the other information given:

<p>If $u_0 = -1$:</p> $\Rightarrow u_0 = l(2)^0 + m(0)(2)^0 = -1$ $\Rightarrow l = -1$	<p>If $u_1 = 8$:</p> $\Rightarrow u_1 = l(2)^1 + m(1)(2)^1 = 8$ $\Rightarrow 2l + 2m = 8$ $\Rightarrow 2(-1) + 2m = 8$ $\Rightarrow m = 5$
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- So, the solution to our difference equation is: $u_n = -1(2)^n + 5n(2)^n$, which can be written in a tidier way if we factorise out the 2^n : $u_n = 2^n(5n - 1)$.

Day 1: Classwork Questions: Pg 156 Ex 8E Qs 1/2/6/8/11/12

Day 2: Classwork Questions: Pg 156/157 Ex 8E Qs 14/16/19/21

- Second Order Inhomogeneous Equations:

- Firstly, a reminder that **inhomogeneous** equations are ones of the form ones $u_{n+2} + u_{n+1} + u_n \pm A = 0$ where A is some other function not related to u_n .
- To solve inhomogeneous equations there are two parts to the solution: the **particular** solution and the **complementary** solution.

- **Example 1:** Pg 160 Ex 8F Q4

Solve the difference equation $T_{n+2} - 4T_{n+1} + 4T_n = 7n - 14$ given $T_0 = 1$ and $T_1 = 15$.

Solution:

- We begin by noticing that the right-hand side of the equation above is of the form $an + b$, so our **particular solution** will be of that form i.e. $T_n = an + b$
- So, $T_{n+1} = a(n+1) + b = an + a + b$ and $T_{n+2} = a(n+2) + b = an + 2a + b$
- Subbing into our initial equation gives:

$$\begin{aligned} T_{n+2} - 4T_{n+1} + 4T_n &= 7n - 14 \\ \Rightarrow an + 2a + b - 4(an + a + b) + 4(an + b) &= 7n - 14 \\ \Rightarrow an - 2a + b &= 7n - 14 \end{aligned}$$

- If we now compare the left-hand side to the right side, it can be seen that $a = 7$ and $-2a + b = -14$.
- Subbing in our value of a gives: $-2(7) + b = -14 \Rightarrow b = 0$
- So, our **particular solution** in this case is: $T_n = an + b = 7n$.
- We now find the **complementary solution** by solving the homogeneous equation $T_{n+2} - 4T_{n+1} + 4T_n = 0$, which we learned how to do in the last section:

$$\begin{aligned} \text{Characteristic Equation to solve: } x^2 - 4x + 4 &= 0 \\ \Rightarrow (x - 2)(x - 2) &= 0 \\ \Rightarrow x - 2 = 0 \quad \text{or} \quad x - 2 = 0 \\ \Rightarrow x = 2 \quad \quad \text{or} \quad x = 2 \end{aligned}$$

- Using Theorem 2 above with our two roots $a = 2$ and $a = 2$, our solution will be in the form:

$$T_n = la^n + mna^n \Rightarrow T_n = l(2)^n + mn(2)^n$$

- To get the **final solution**, we add the particular solution and the complementary solution: $T_n = l(2)^n + mn(2)^n + 7n$

If $T_0 = 1$: $\Rightarrow T_0 = l(2)^0 + m(0)(2)^0 + 7(0) = 1$ $\Rightarrow l = 1$	If $T_1 = 15$: $\Rightarrow T_1 = l(2)^1 + m(1)(2)^1 + 7(1) = 15$ $\Rightarrow 2l + 2m + 7 = 15$ $\Rightarrow 2(1) + 2m + 7 = 15$ $\Rightarrow 2m = 6$ $\Rightarrow m = 3$
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- So, the **final solution** to our difference equation is: $T_n = 1(2)^n + 3n(2)^n + 7n$, which can be written in a tidier way if we factorise out the 2^n : $u_n = 2^n(3n + 1) + 7n$.

Classwork Questions: Pg 160 Ex 8F Qs 1/2/7

- Example 2:**

Solve the difference equation $2u_{n+2} - u_{n+1} - 3u_n = 3^n$ given $u_0 = 3$ and $u_1 = -5$.

Solution:

- This time the right-hand side is in the form $k \cdot 3^n$ so our solution will be of that form i.e.

$$u_n = k \cdot 3^n.$$

- So, $u_{n+1} = k \cdot 3^{(n+1)} = k \cdot 3^n \cdot 3^1 = 3k \cdot 3^n$ and $u_{n+2} = k \cdot 3^{(n+2)} = k \cdot 3^n \cdot 3^2 = 9k \cdot 3^n$.

- Subbing these expressions into the equation we were asked to solve initially gives:

$$2u_{n+2} - u_{n+1} - 3u_n = 3^n$$

$$\Rightarrow 2(9k \cdot 3^n) - (3k \cdot 3^n) - 3(k \cdot 3^n) = 3^n$$

- If we factorise out the 3^n on the left-hand side:

$$3^n(18k - 3k - 3k) = 3^n$$

$$\Rightarrow 3^n(12k) = 3^n$$

- For this to be true $12k$ must be equal to 1

$$\Rightarrow 12k = 1$$

$$\Rightarrow k = \frac{1}{12}$$

- So, our particular solution is: $u_n = \frac{1}{12} \cdot 3^n$.

- As in example 1, we now must find the complementary solution by solving the associated characteristic equation:

$$2x^2 - x + 3 = 0$$

$$\Rightarrow (2x - 3)(x + 1) = 0$$

$$\Rightarrow x = \frac{3}{2} \quad \text{or} \quad x = -1$$

- As we have two roots our complementary solution will be in the form $u_n = la^n + mb^n$ where $a = \frac{3}{2}$ and $b = -1$: $\Rightarrow u_n = l\left(\frac{3}{2}\right)^n + m(-1)^n$

- We now combine the particular and complementary solutions again to get the full solution:

$$u_n = l\left(\frac{3}{2}\right)^n + m(-1)^n + \frac{1}{12} \cdot 3^n$$

- Using the other information given in the question to find l and m :

If $u_0 = 3$:

$$\Rightarrow u_0 = l\left(\frac{3}{2}\right)^0 + m(-1)^0 + \frac{1}{12} \cdot 3^0 = 3$$

$$\Rightarrow l + m = 3 - \frac{1}{12}$$

$$\Rightarrow 12l + 12m = 35$$

If $u_1 = -5$:

$$\Rightarrow u_1 = l\left(\frac{3}{2}\right)^1 + m(-1)^1 + \frac{1}{12} \cdot 3^1 = -5$$

$$\Rightarrow \frac{3}{2}l - m + \frac{1}{4} = -5$$

$$\Rightarrow 6l - 4m + 1 = -20$$

$$\Rightarrow 6l - 4m = -21$$

- Solving these two simultaneous equations gives: $l = -\frac{14}{15}$ and $m = \frac{77}{20}$, so our final solution

will be: $u_n = \left(-\frac{14}{15}\right)\left(\frac{3}{2}\right)^n + \left(\frac{77}{20}\right)(-1)^n + \frac{1}{12} \cdot 3^n$

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Day 2: Classwork Questions: Pg 160 Ex 8F Qs 14/15/16

Revision Questions and Test