Topic 7: Functions/Graphs

1) The Basics:

a) Terminology:

- **Domain** = the values that are put **into** a function.
- **Range** = the values that come **out** of a function.
- **Codomain** = the values that **could come out** of a function.

b) Notation:

The different ways functions are written are:

- $f(x) = x^2 + 3x$
- $f:x \rightarrow x^2 + 3x$
- $y = x^2 + 3x$

c) Evaluating Functions:

Example: If $f(x) = 2x^2 + 3$, find $f(3)$ and $f(-1)$.

$f(3) = 2(3)^2 + 3 = 21$

$f(-1) = 2(-1)^2 + 3 = 5$

d) Finding Inputs of Functions:

Example: If $f(x) = 5x - 3$, find the value of x for which $f(x) = 12$.

$f(x) = 12$

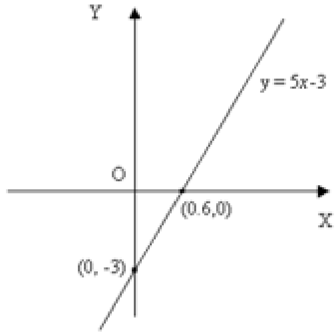
$\Rightarrow 5x - 3 = 12$

$\Rightarrow 5x = 15$

$\Rightarrow x = 3$

2) Types of Graphs:

a) Linear: $y = ax + b$

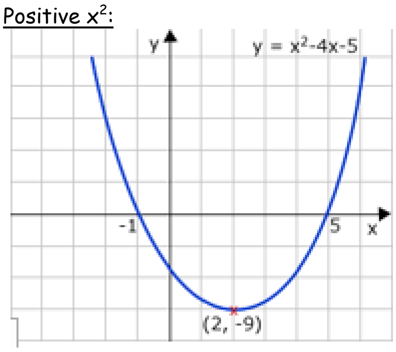


Notes:

- Graph above is for function of the form $y = ax + b$
- If 'a' is positive, the line increases from left to right but if 'a' is negative, the line decreases from left to right
- Function of the form $y = ax$ would be a line through the origin 'O'
- The root is where the graph crosses the x-axis….in the graph above, the root is 0.6.

b) Quadratic: $y = ax^2 + bx + c$

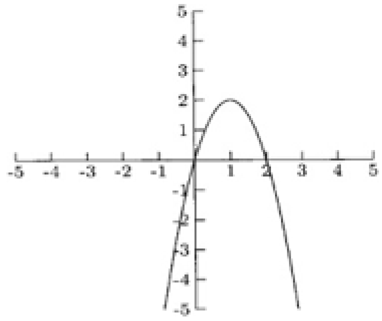
Positive x^2:



Notes:

- Graph above is for function of the form $y = ax^2 + bx + c$, where 'a' is a positive number
- Roots are where the graph crosses the x-axis…..in the graph above, the roots are -1 and 5
- The minimum point is the lowest point on the graph….in the graph above the minimum point is (2,-9).

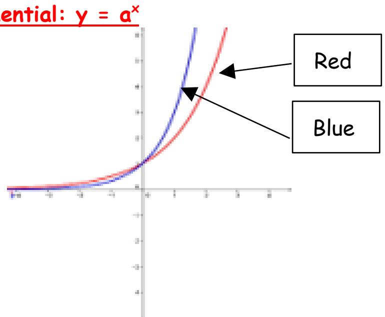
Negative x^2:



Notes:

- Graph above is for function of the form $y = ax^2 + bx + c$, where 'a' is a negative number
- Roots are where the graph crosses the x-axis…..in the graph above, the roots are 0 and 2
- The maximum point is the highest point on the graph…..in the graph above the maximum point is (1, 2)

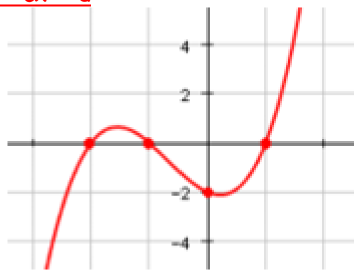
c) Exponential: $y = a^x$



Notes:

- Red line in the graph above is graph of $y = 2^x$ and blue line is graph of $y = 3^x$
- All graphs of the form a^x pass through the point (0,1)
- Note that $y = 3^x$ rises at a steeper rate after passing through the point (0,1)
- Graphs of the form $a2^x$ would look similar to the graph of 2^x but would be shifted on the left

d) Cubic: $y = ax^3 + bx^2 + cx + d$



Notes:

- Graph above is for function of the form $y = ax^3 + bx^2 + cx + d$, where 'a' is a positive number
- If 'a' was negative the S shape would be the other way around i.e. coming in from the top left and leaving in the bottom right of the graph above
- Roots are where the graph crosses the x-axis, so cubic graphs have three roots
- The local minimum point is the point at the base of the trough in the graph….in the graph above the local minimum point is (0.4, -2)
- The local maximum is the peak of the hill in the graph…..in the graph above the local maximum point is (-1.5, 0.8)

3) Drawing/Interpreting Graphs:

a) Drawing Graphs:

- Just fill in the values from the domain and use calculator.

Example: Draw the graph of $x^2 - 3x - 4$, in the domain $-2 \leq x \leq 1$

$f(x) = x^2 - 3x - 4$

$f(-2) = (-2)^2 - 3(-2) - 4 = 6$ $(-2, 6)$
$f(-1) = (-1)^2 - 3(-1) - 4 = 0$ $(-1, 0)$
$f(0) = (0)^2 - 3(0) - 4 = -4$ $(0, -4)$
$f(1) = (1)^2 - 3(1) - 4 = -6$ $(1, -6)$
$f(2) = (2)^2 - 3(2) - 4 = -6$ $(2, -6)$
$f(3) = (3)^2 - 3(3) - 4 = -4$ $(3, -4)$

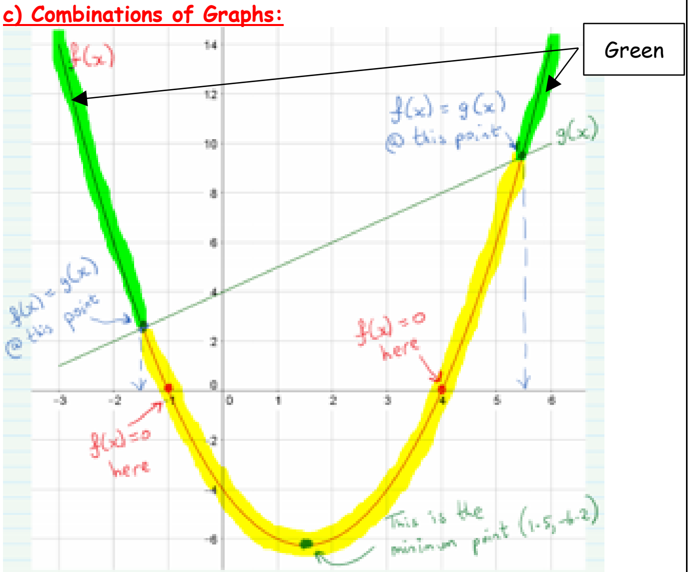
Can plot these on graph paper. Should know shape of graph from Section 2.

b) Interpreting Graphs:

Tip:
Use ruler and dotted lines when working out values from a graph

- To find f(2) or f(-1) from graph, for example: come up from x = 2 or x = -1 until you hit the graph and then go across to y value
- To find f(x) = 3 or f(x) = -2 from graph: draw a line through y = 3 or y = -2, and then come up/down to x-axis from the point(s) where the line crosses the graph
- Roots are where graph crosses x-axis i.e. f(x) = 0
- Axis of symmetry is the line that cuts the graph into 2. Only arises in U or ∩ shape.

c) Combinations of Graphs:



- f(x) = g(x) is where the two functions intersect
- Yellow Highlighted section is where f(x) < g(x)
- Green highlighted section is where f(x) > g(x)

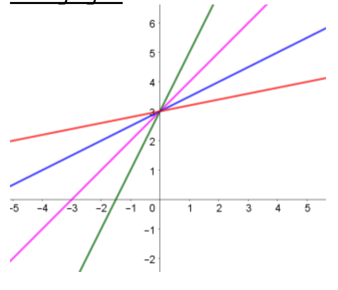
4) Graph Transformations:

a) Linear Graphs (y = ax + b):

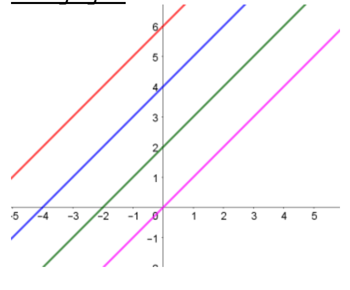
Notes:

- Changing the **'b'** changes the **y-intercept** to whatever the value of b is.
- Changing the **'a'** changes the **slope** of the graph to whatever the value of a is.

Changing 'a'



Changing 'b'

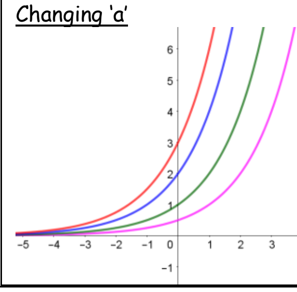


b) Exponential Graphs ($y = ak^x + b$):

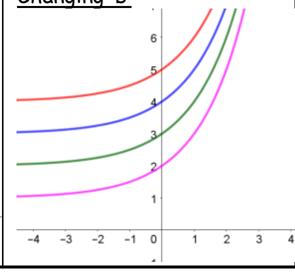
Notes:

- Changing the **'a'** in $y = ak^x$ changes the **y-intercept**. In a graph of the form $y = ak^x$, the graph crosses the y-axis at (0, a).
- Changing the **'b'** in $y = ak^x + b$ **shifts** the whole graph **up or down** crosses the y-axis at (0, b + 1). If b is positive it moves up and if it's negative it moves down.

Changing 'a'



Changing 'b'

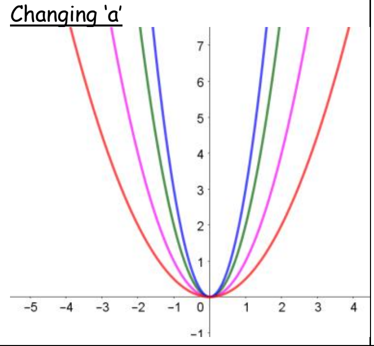


c) Quadratic Graphs ($y = ax^2 + bx + c$):

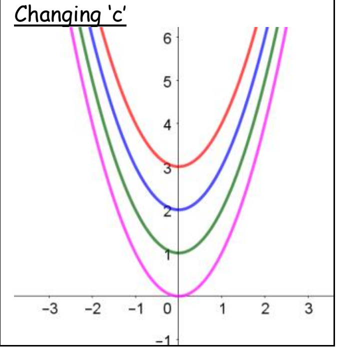
Notes:

- Changing the **'a' narrows** the **U shape**. The bigger the value of a the narrower it gets.
- Changing the **'c' shifts** the whole graph **up or down** depending on the value of c. If c is positive it moves up and if it's negative it moves down.

Changing 'a'



Changing 'c'

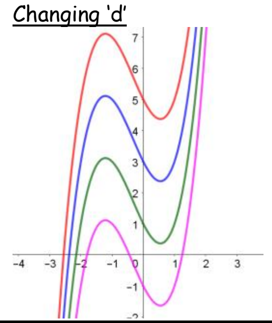


d) Cubic Graphs ($y = ax^3 + bx^2 + cx + d$):

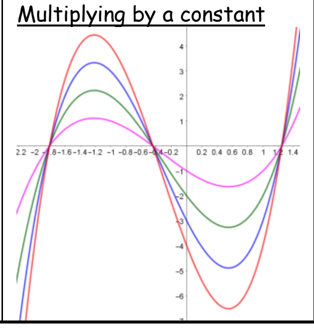
Notes:

- Changing the **'d' shifts** the whole graph **up or down** depending on the value of d. If d is positive it moves up and if it's negative it moves down.
- If we **multiply** the entire function **by a constant** (e.g. 2) then the **max** and **min points** will be **twice** as high and low.

Changing 'd'



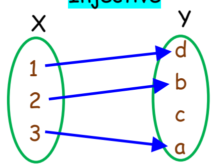
Multiplying by a constant



5) Injective/Surjective/Bijective Functions:

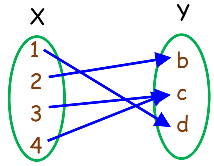
a) Injective/Surjective/Bijective Functions:

Injective



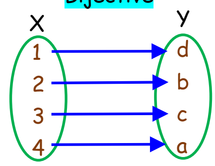
A function is injective (or 1-1, pronounced "1 to 1") if every output has a unique input i.e. there can't be more than one input going to a particular output.

Surjective



A function is surjective (or "onto") every element in the codomain is an output i.e. all the elements in Y have an input.

Bijective

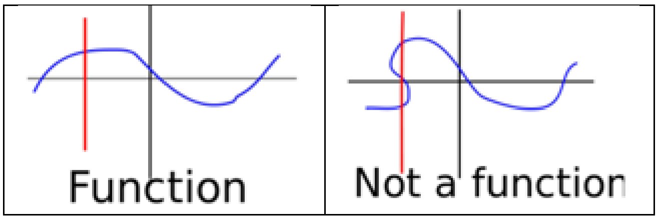


A function is bijective if it is both injective and surjective i.e. every output has a unique input and all outputs have an input.

b) Vertical Line Test: (Is a mapping a function or not?)

Notes:

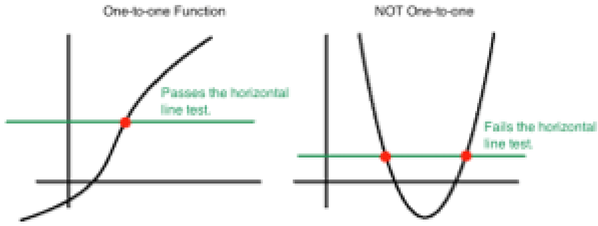
- As a function has a unique output for every input, a vertical line drawn through the graph should only intersect the graph once.
- If it's possible to draw a vertical line that **crosses a graph more than once**, then the mapping is **not** a function



c) Horizontal Line Test: (Is function injective/surjective or not?)

Notes:

- As an **injective** (one-to-one) function is one where every output has a unique input, we can use a horizontal line to check if a function is injective, or not.
  - If a horizontal line can cut a graph **more than once**, then the function is **not injective**.
- In a **surjective** function, all the outputs have an input, so **every** horizontal line y = b, intersects the graph of the function **at least** one point.
  - If it's possible to draw a horizontal line in the **codomain** that **doesn't cross** the graph, then the function is **surjective**.



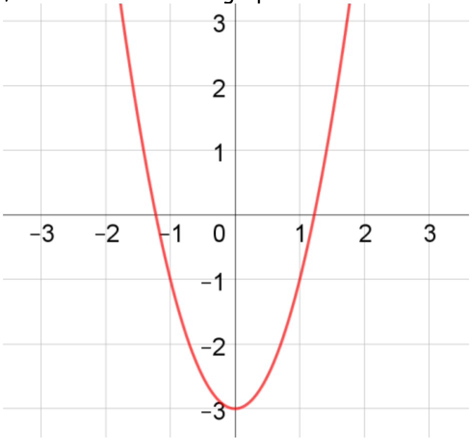
Example 1: Function $h(x) = 2x^2 - 3$ is defined for the domain $x \in R$.

i) Find the range of the function.

ii) Determine if the function is injective, or not.

iii) Determine if the function is surjective, or not.

- First, let's have a look at a graph of $2x^2 - 3$:



i) As the domain of this function is any real number, and as we can see from the graph, the range of the function can be any value above -3: => **Range = [-3, ∞)**

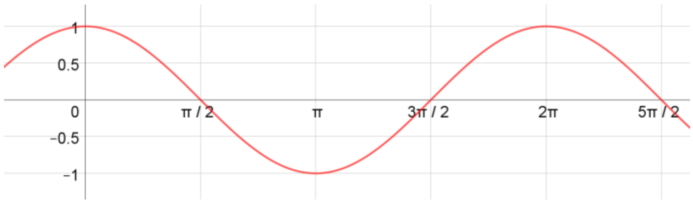
ii) If we draw a horizontal line through the graph, it will cross at two points (with the exception of the horizontal line through the minimum point) => **Function is not injective**

iii) If we draw a horizontal line through the graph at y = -4, then the line will not intersect the graph at any point => **Function is not surjective**

Example 2: Consider the function f(x) = cos(x) where f:R -> [-1, 1].

i) Determine whether the function is surjective or not.

ii) Determine whether the function is injective or not.



i)

- A horizontal line drawn through any y value bigger than 1 or less than -1, will not cut this graph, so normally the graph of cos(x) couldn't be surjective.

- In this question, however, f:R -> [-1, 1] tells us that this particular function maps all Real values to the codomain [-1, 1]

=> if we are restricted to the codomain of [-1, 1] then a horizontal line will always cut the graph at at least one point

=> the function is surjective

ii)

- We can clearly draw a horizontal line on the graph above that will cut the graph at more than one place e.g. y = 0.5

=> the function is not injective

6) Composite Functions:

Notes:

- Two or more functions together = **composite functions**.
- Symbol: 'o'
- Function **written second** must be **used first**.
  E.g. f(g(x)) means you use g(x) first, and then put output into f
- Composition written as f o g(x) or (f o g)(x) or f(g(x)) or fg(x)
- Composition is not commutative i.e. f o g(x) ≠ g o f(x)

Example 1: f(x) = 2x + 1 and g(x) = x^2 + 3 are two functions.

i) Find the value of f o g(3).

ii) Find the value of gf(3)

i) f o g(3) means we have to get g(3) first:

g(3) = $(3)^2$ + 3 = 12

- We now have to put 12 into the f function:

f(12) = 2(12) + 1 = 24 + 1 = 25

ii) gf(3) means we have to get f(3) first:

f(3) = 2(3) + 1
= 6 + 1 = 7

- We now have to put 7 into the g function:

g(7) = $(7)^2$ + 3
= 49 + 3 = 52

Example 2: Decompose h(t) = $(2t - 3)^2$ into two separate functions.

- In the function h, the input has to first be multiplied by 2, and then 3 has to be taken from it, so we could define a function p(t) = 2t – 3 to represent that.
- Output has to be squared now, so define a function q(t) = t^2 to do that.
- Now, put the p and q functions together (take care with order).
- We want to do p first, so we write: h = q o p(t)

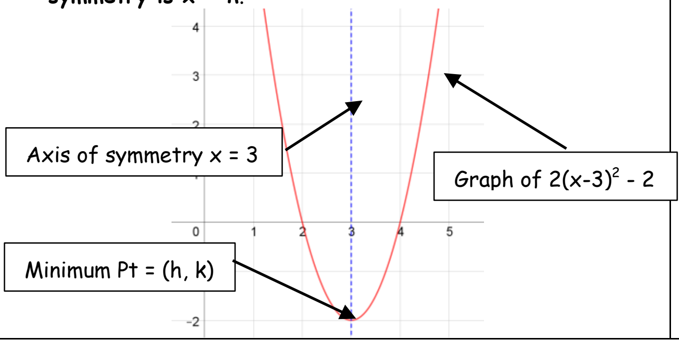
7) Completed square form:

Notes:

- To be in completed square form:

A function $ax^2 + bx + c$ is written in the form $a(x - h)^2 + k$.

- If a quadratic is in completed square form, **axis of symmetry is x = h.**



Example: Write the quadratic function $x^2 - 6x + 13$ in completed square form.

Step 1: First, write an expression of the form $(x + a)^2$ that will give the first two terms of the function:
=> $(x - 3)^2$ would give us x^2 and -6x when multiplied out (as well as a 9)

Step 2: Now we add/subtract a value from our expression, to get back the original function:
=> $(x - 3)^2$ is $x^2 - 6x + 9$ when multiplied out fully, so we have to add on a 4 in this case to get back the original function $x^2 - 6x + 13$
=> the completed square form of $x^2 - 6x + 13$ is $(x - 3)^2 + 4$.

- This function would have an axis of symmetry of x = 3 and a minimum point of (3, 4).

8) Inverse Functions:

a) Inverse Functions:

Notes:

- When a number passes two different functions, and we get original number back, then functions are said to be **inverses**.
- If f o g(x) = g o f(x) = x, then f and g are inverses.
- Two functions that would be inverses of each other would be f(x) = 2x + 1 and g(x) = $\frac{x-1}{2}$ because:
  f(1) = 2(1) + 1 = 3 and then g(3) = $\frac{3-1}{2}$ = 1
- Notation $\mathbf{f^{-1}}$ to symbolise the inverse of the function f.
- Only **bijective** functions **have inverses**.

Steps to find inverse functions:

1. Write the function in the form y in terms of x.
2. Rearrange the function to write x in terms of y.
3. Swap the x and y.

Example 1: Given a function f(x) = $\frac{4-3x}{8}, x \in R$:

i) Find the inverse of the function, f^{-1}.

ii) Verify that the inverse is, in fact, the inverse.

$$y = \frac{4 - 3x}{8}$$
=> $8y = 4 - 3x$
=> $3x = 4 - 8y$
=> $x = \frac{4-8y}{3}$
=> $f^{-1}(x) = \frac{4-8x}{3}, x \in R$

ii) To do this part we try filling in a value to f and then to g and see what happens:

$$f(4) = \frac{4 - 3(4)}{8} = \frac{4-12}{8} = \frac{-8}{8} = -1$$
$$g(-1) = \frac{4 - 8(-1)}{3} = \frac{4+8}{3} = \frac{12}{3} = 4$$

=> we got back original value we put in => f and g are inverses

b) Graphs of Inverse Functions:

Notes:

- Red graph below = graph of f, blue graph = graph of f^{-1}.
- If (x, y) is on a graph of f, then (y, x) will be on f^{-1}.
- f and f^{-1}, are images of each other under **reflection (axial symmetry) in the line x = y** (shown in green below).