Worked Solutions

Q1.

i) $u_n = 3u_{n-1} + 5 \quad u_0 = 2$

$\underline{n=1}$

$u_1 = 3u_0 + 5$
$= 3(2) + 5$

$\underline{n=2}$

$u_2 = 3u_1 + 5$
$= 3(3(2)+5)+5$
$= (3)^2(2) + 3(5) + 5$

$\underline{n=3}$

$u_3 = 3u_2 + 5$
$= 3((3)^2(2) + 3(5) + 5) + 5$
$= (3)^3(2) + \underbrace{(3)^2(5) + 3(5) + 5}_{a=5 \quad r=3}$

$$\Rightarrow u_n = (3)^n(2) + \frac{5(1-(3)^n)}{1-3}$$
$$= 2(3)^n - \frac{5}{2}(1-(3)^n)$$
$$= 2(3)^n - \frac{5}{2} + \frac{5}{2}(3)^n$$
$$= \boxed{\frac{9}{2}(3)^n - \frac{5}{2}}$$

ii) $u_n = \frac{1}{4}u_{n-1} + 2, \quad u_0 = 12$

$\underline{n=1}$

$u_1 = \frac{1}{4}u_0 + 2$
$= \frac{1}{4}(12) + 2$

$\underline{n=2}$

$u_2 = \frac{1}{4}(u_1) + 2$
$= \frac{1}{4}\left(\frac{1}{4}(12)+2\right)+2$
$= \left(\frac{1}{4}\right)^2(12) + \frac{1}{4}(2) + 2$

$\underline{n=3}$

$u_3 = \frac{1}{4}u_2 + 2$
$= \frac{1}{4}\left(\left(\frac{1}{4}\right)^2(12) + \frac{1}{4}(2) + 2\right) + 2$
$= \left(\frac{1}{4}\right)^3(12) + \underbrace{\left(\frac{1}{4}\right)^2(2) + \frac{1}{4}(2) + 2}_{a=2 \quad r=\frac{1}{4}}$

$$\Rightarrow u_n = \left(\frac{1}{4}\right)^n(12) + \frac{2\left(1-\left(\frac{1}{4}\right)^n\right)}{1-\frac{1}{4}}$$
$$= 12\left(\frac{1}{4}\right)^n + \frac{8}{3}\left(1-\left(\frac{1}{4}\right)^n\right)$$
$$= \left(12-\frac{8}{3}\right)\left(\frac{1}{4}\right)^n + \frac{8}{3}$$
$$= \boxed{\frac{28}{3}\left(\frac{1}{4}\right)^n + \frac{8}{3}}$$

iii) $u_n = -2u_{n-1} + 7 \qquad u_1 = 8$

$\underline{n=1}$

$u_1 = -2u_0 + 7$

$8 = -2u_0 + 7$

$2u_0 = -1$

$u_0 = \frac{-1}{2}$

$\Rightarrow u_1 = -2\left(\frac{-1}{2}\right) + 7$

$\underline{n=2}$

$u_2 = -2u_1 + 7$

$= -2\left(-2\left(\frac{-1}{2}\right) + 7\right) + 7$

$= (-2)^2\left(\frac{-1}{2}\right) - 2(7) + 7$

$\underline{n=3}$

$u_3 = -2u_2 + 7$

$= -2\left((-2)^2\left(\frac{-1}{2}\right) - 2(7) + 7\right) + 7$

$= (-2)^3\left(\frac{-1}{2}\right) + \underbrace{(2)^2(7) - 2(7) + 7}$

$a = 7 \quad r = -2$

$$\Rightarrow u_n = (-2)^n\left(-\frac{1}{2}\right) + \frac{7\left(1 - (-2)^n\right)}{1 - (-2)}$$

$$= \frac{-1}{2}(-2)^n + \frac{7}{3}\left(1 - (-2)^n\right)$$

$$= \left(\frac{-1}{2} - \frac{7}{3}\right)(-2)^n + \frac{7}{3}$$

$$= \boxed{\frac{-17}{6}(-2)^n + \frac{7}{3}}$$

iv) $u_n = 7u_{n-1} - 30, \quad u_0 = 4$

$\underline{n=1}$

$u_1 = 7u_0 - 30$

$= 7(4) - 30$

$\underline{n=2}$

$u_2 = 7u_1 - 30$

$= 7\left(7(4) - 30\right) - 30$

$= (7)^2(4) - 7(30) - 30$

$\underline{n=3}$

$u_3 = 7u_2 - 30$

$= 7\left((7)^2(4) - 7(30) - 30\right) - 30$

$= (7)^3(4) - (7)^2(30) - 7(30) - 30$

$= (7)^3(4) - \left[30 + (30)(7) + (7)^2(30)\right]$

$a = 30 \quad r = 7$

$$\Rightarrow u_n = (7)^n(4) - \frac{30\left(1 - (7)^n\right)}{1 - 7}$$

$$= (7)^n(4) + 5\left(1 - (7)^n\right)$$

$$= (4 - 5)(7)^n + 5$$

$$= \boxed{-1(7)^n + 5}$$

v) $u_n = -\frac{1}{2} u_{n-1} + 3$, $u_1 = 10$

$\underline{n=1}$

$u_1 = -\frac{1}{2} u_0 + 3$

$10 = -\frac{1}{2} u_0 + 3$

$\frac{1}{2} u_0 = -7$

$u_0 = -14$

$\Rightarrow u_1 = -\frac{1}{2}(-14) + 3$

$\underline{n=2}$

$u_2 = -\frac{1}{2} u_1 + 3$

$= -\frac{1}{2}\left(-\frac{1}{2}(-14) + 3\right) + 3$

$= \left(-\frac{1}{2}\right)^2(-14) - \frac{1}{2}(3) + 3$

$\underline{n=3}$

$u_3 = -\frac{1}{2} u_2 + 3$

$= -\frac{1}{2}\left(\left(-\frac{1}{2}\right)^2(-14) - \frac{1}{2}(3) + 3\right) + 3$

$= \left(-\frac{1}{2}\right)^3(-14) + \underbrace{\left(\frac{1}{2}\right)^2(3) - \frac{1}{2}(3) + 3}$

$a = 3 \quad r = -\frac{1}{2}$

$$\Rightarrow u_n = \left(-\frac{1}{2}\right)^n(-14) + \frac{3\left(1 - \left(-\frac{1}{2}\right)^n\right)}{1 - \left(-\frac{1}{2}\right)}$$

$$= \left(-\frac{1}{2}\right)^n(-14) + 2\left(1 - \left(-\frac{1}{2}\right)^n\right)$$

$$= (-14 - 2)\left(-\frac{1}{2}\right)^n + 2$$

$$= \boxed{-16\left(-\frac{1}{2}\right)^n + 2}$$

vi) $u_n = \frac{2}{3} u_{n-1} - 1$, $u_0 = 18$

$\underline{n=1}$

$u_1 = \frac{2}{3} u_0 - 1$

$= \frac{2}{3}(18) - 1$

$\underline{n=2}$

$u_2 = \frac{2}{3} u_1 - 1$

$= \frac{2}{3}\left(\frac{2}{3}(18) - 1\right) - 1$

$= \left(\frac{2}{3}\right)^2(18) - \frac{2}{3}(1) - 1$

$\underline{n=3}$

$u_3 = \frac{2}{3} u_2 - 1$

$= \frac{2}{3}\left(\left(\frac{2}{3}\right)^2(18) - \frac{2}{3}(1) - 1\right) - 1$

$= \left(\frac{2}{3}\right)^3(18) - \left(\frac{2}{3}\right)^2(1) - \frac{2}{3} - 1$

$= \left(\frac{2}{3}\right)^3(18) - \left[1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2(1)\right]$

$a = 1 \quad r = \frac{2}{3}$

$$\Rightarrow u_n = \left(\frac{2}{3}\right)^n(18) - \frac{1\left(1 - \left(\frac{2}{3}\right)^n\right)}{1 - \frac{2}{3}}$$

$$= \left(\frac{2}{3}\right)^n(18) - 3\left(1 - \left(\frac{2}{3}\right)^n\right)$$

$$= (18 + 3)\left(\frac{2}{3}\right)^n - 3$$

$$= \boxed{21\left(\frac{2}{3}\right)^n - 3}$$

Q2.

i) $u_n = \frac{2}{3}u_{n-1} + 9 \qquad u_0 = 162$

$\underline{n=1}$

$u_1 = \frac{2}{3}u_0 + 9$

$= \frac{2}{3}(162) + 9$

$\underline{n=2}$

$u_2 = \frac{2}{3}u_1 + 9$

$= \frac{2}{3}\left(\frac{2}{3}(162) + 9\right) + 9$

$= \left(\frac{2}{3}\right)^2(162) + \frac{2}{3}(9) + 9$

$\underline{n=3}$

$u_3 = \frac{2}{3}u_2 + 9$

$= \frac{2}{3}\left(\left(\frac{2}{3}\right)^2(162) + \frac{2}{3}(9) + 9\right) + 9$

$= \left(\frac{2}{3}\right)^3(162) + \underbrace{\left(\frac{2}{3}\right)^2(9) + \frac{2}{3}(9) + 9}_{a=9 \quad r=2/3}$

$$\Rightarrow u_n = \left(\frac{2}{3}\right)^n(162) + \frac{9\left(1-\left(\frac{2}{3}\right)^n\right)}{1-\frac{2}{3}}$$

$$= \left(\frac{2}{3}\right)^n(162) + 27\left(1 - \left(\frac{2}{3}\right)^n\right)$$

$$= (162-27)\left(\frac{2}{3}\right)^n + 27$$

$$= \boxed{135\left(\frac{2}{3}\right)^n + 27}$$

ii) $u_8 = 135\left(\frac{2}{3}\right)^8 + 27$

$= \boxed{\frac{7841}{243}}$

iii) $\lim_{n\to\infty} 135\left(\frac{2}{3}\right)^n + 27$

$= 135(0) + 27$

$= \boxed{27}$

Q3.

i) $u_n = \frac{1}{4}u_{n-1} + 90, \quad u_1 = 800$

$\underline{n=1}$

$u_1 = \frac{1}{4}(u_0) + 90$

$800 = \frac{1}{4}(u_0) + 90$

$\frac{1}{4}u_0 = 710$

$u_0 = 2840$

$\Rightarrow u_1 = \frac{1}{4}(2840) + 90$

$\underline{n=2}$

$u_2 = \frac{1}{4}u_1 + 90$

$= \frac{1}{4}\left(\frac{1}{4}(2840)+90\right)+90$

$= \left(\frac{1}{4}\right)^2(2840) + \frac{1}{4}(90) + 90$

$\underline{n=3}$

$u_3 = \frac{1}{4}u_2 + 90$

$= \left(\frac{1}{4}\right)^3(2840) + \underbrace{\left(\frac{1}{4}\right)^2(90) + \frac{1}{4}(90) + 90}_{a=90 \quad r=1/4}$

$$\Rightarrow u_n = \left(\frac{1}{4}\right)^n(2840) + \frac{90\left(1-\left(\frac{1}{4}\right)^n\right)}{1-1/4}$$

$$= \left(\frac{1}{4}\right)^n(2840) + 120\left(1-\left(\frac{1}{4}\right)^n\right)$$

$$= (2840-120)\left(\frac{1}{4}\right)^n + 120$$

$$= \boxed{2720\left(\frac{1}{4}\right)^n + 120}$$

ii) 6^{th} term starting @ $u_1 = u_6$

$$u_6 = 2720\left(\frac{1}{4}\right)^6 + 120$$

$$= \boxed{\frac{15445}{128}} \text{ or } \boxed{120.664}$$

iii) $2720\left(\frac{1}{4}\right)^n + 120 < 125$

$2720\left(\frac{1}{4}\right)^n < 5$

$\left(\frac{1}{4}\right)^n < \frac{5}{2720}$

$\log\left(\frac{1}{4}\right)^n < \log\left(\frac{5}{2720}\right)$

$n\log\left(\frac{1}{4}\right) < \log\left(\frac{5}{2720}\right)$

$\Rightarrow n > \frac{\log(5/2720)}{\log(1/4)}$ (as $\log\left(\frac{1}{4}\right) < 0$)

$n > 4.54 \Rightarrow \boxed{n=5}$

OR Just work out terms:

$u_1 = 800 \quad u_2 = 290 \quad u_3 = 162.5$

$u_4 = 130.625 \quad u_5 = 122.656$

iv) $\lim_{n\to\infty} 2720\left(\frac{1}{4}\right)^n + 120$

$= 2720(0) + 120 = \boxed{120}$

Q4.

(i) $u_{n+2} + 7u_{n+1} = -12u_n \qquad u_0 = 2 \qquad u_1 = -9$

$u_{n+2} + 7u_{n+1} + 12u_n = 0$

Char eqn:

$x^2 + 7x + 12 = 0$

$(x+4)(x+3) = 0$

$x+4 = 0 \quad \text{or} \quad x+3 = 0$

$x = -4 \qquad x = -3$

Theorem:

$u_n = l(-4)^n + m(-3)^n$

<u>$u_0 = 2$</u>

$l(-4)^0 + m(-3)^0 = 2$

$l + m = 2 \quad : \text{I}$

<u>$u_1 = -9$</u>

$l(-4)^1 + m(-3)^1 = -9$

$-4l - 3m = -9$

$4l + 3m = 9 \quad : \text{II}$

<u>Solving I & II:</u>

$\text{I} \times 3: \quad 3l + \cancel{3m} = 6$

$\text{II}: \quad (-)4l \overset{(-)}{+} \cancel{3m} = (-)9$

$-l = -3$

$l = 3$

<u>Using I:</u> $3 + m = 2$

$m = -1$

$\Rightarrow \quad u_n = \boxed{3(-4)^n - 1(-3)^n}$

(ii) $2T_n - 5T_{n-1} = 12T_{n-2}$ $\quad u_1 = 1 \quad u_2 = 20$

$2T_n - 5T_{n-1} - 12T_{n-2} = 0$

Char eqn:

$2x^2 - 5x - 12 = 0$

$(2x+3)(x-4) = 0$

$2x + 3 = 0$ or $x - 4 = 0$

$2x = -3$ $\quad x = 4$

$x = -\frac{3}{2}$

Theorem:

$$u_n = l\left(-\frac{3}{2}\right)^n + m(4)^n$$

$\underline{u_1 = 1}$

$l\left(-\frac{3}{2}\right)^1 + m(4)^1 = 1$

$-3l + 8m = 2$: I

$\underline{u_2 = 20}$

$l\left(-\frac{3}{2}\right)^2 + m(4)^2 = 20$

$l\left(\frac{9}{4}\right) + 16m = 20$

$9l + 64m = 80$: II

<u>Solving I & II:</u>

I×8: $-24l + \cancel{64m} = 16$

II: $(-)9l \overset{(-)}{+} \cancel{64m} \overset{(-)}{=} 80$

$-33l = -64$

$l = \frac{64}{33}$

<u>Using I:</u> $-3\left(\frac{64}{33}\right) + 8m = 2$

$-\frac{64}{11} + 8m = 2$

$8m = 2 + \frac{64}{11}$

$8m = \frac{86}{11}$

$m = \frac{43}{44}$

$$\Rightarrow u_n = \boxed{\frac{64}{33}\left(-\frac{3}{2}\right)^n + \frac{43}{44}(4)^n}$$

(iii) $2T_{n+1} + T_{n+2} = 15T_n$, $T_1 = 1$ $T_2 = 14$

$T_{n+2} + 2T_{n+1} - 15T_n = 0$

Char eqn:

$x^2 + 2x - 15 = 0$

$(x+5)(x-3) = 0$

$x+5 = 0$ or $x-3 = 0$

$x = -5$ $\quad x = 3$

Theorem:

$T_n = l(-5)^n + m(3)^n$

$T_1 = 1$

$l(-5)^1 + m(3)^1 = 1$

$-5l + 3m = 1$: I

$T_2 = 14$

$l(-5)^2 + m(3)^2 = 14$

$25l + 9m = 14$: II

Solving I & II:

I×3: $-15l + \cancel{9m} = 3$

II: $(-)25l \overset{(-)}{+} \cancel{9m} \overset{(-)}{=} 14$

$-40l = -11$

$l = \frac{11}{40}$

Using I: $-5\left(\frac{11}{40}\right) + 3m = 1$

$-\frac{11}{8} + 3m = 1$

$3m = 1 + \frac{11}{8}$

$3m = \frac{19}{8}$

$m = \frac{19}{24}$

$\Rightarrow T_n = \boxed{\frac{11}{40}(-5)^n + \frac{19}{24}(3)^n}$

(iv) $u_{n+2} - 6u_{n+1} = -9u_n$, $u_0 = 3$ $u_1 = 6$

$u_{n+2} - 6u_{n+1} + 9u_n = 0$

Char eqn:

$x^2 - 6x + 9 = 0$

$(x-3)(x-3) = 0$

$x - 3 = 0$ or $x - 3 = 0$

$x = 3$ $x = 3$

Theorem:

$u_n = l(3)^n + mn(3)^n$

$\underline{u_0 = 2}$

$l(3)^0 + m(0)(3)^0 = 3$

$l = 3$

$\underline{u_1 = 6}$

$l(3)^1 + m(1)(3)^1 = 6$

$3l + 3m = 6$

$3(3) + 3m = 6$

$9 + 3m = 6$

$3m = -3$

$m = -1$

$\Rightarrow u_n = 3(3)^n - n(3)^n$

$= \boxed{(3-n)(3)^n}$

Q5.

i) $u_n - 3u_{n-1} = n-5 \qquad u_0 = 1$

Complementary Soln:

Char eqn: $x^2 - 3x = 0$

$x(x-3) = 0$

$x = 0 \quad \text{or} \quad x - 3 = 0$

$x = 3$

$\Rightarrow u_n = l(0)^n + m(3)^n$

$= m(3)^n$

Particular Soln:

$n-5$ is linear $\Rightarrow u_n = an + b$

$\Rightarrow u_{n-1} = a(n-1) + b$

$= an - a + b$

$u_n - 3u_{n-1} = n-5$

$an + b - 3(an - a + b) = n - 5$

$an + b - 3an + 3a - 3b = n - 5$

$-2an + 3a - 2b = n - 5$

$\Rightarrow -2a = 1 \qquad 3a - 2b = -5$

$a = -\frac{1}{2} \qquad 3(-\frac{1}{2}) - 2b = -5$

$-\frac{3}{2} - 2b = -5$

$-3 - 6b = -10$

$-6b = -7$

$b = \frac{-7}{-6} = \frac{7}{6}$

$\Rightarrow$ Total Soln $= m(3)^n - \frac{1}{2}n + \frac{7}{6}$

$u_0 = 1$

$m(3)^0 - \frac{1}{2}(0) + \frac{7}{6} = 1$

$m + \frac{7}{6} = 1$

$m = 1 - \frac{7}{6} = -\frac{1}{6} \Rightarrow u_n = \boxed{-\frac{1}{6}(3)^n - \frac{1}{2}n + \frac{7}{6}}$

ii) $u_9 = -\frac{1}{6}(3)^9 - \frac{1}{2}(9) + \frac{7}{6} = \boxed{\frac{-19703}{6}}$ or $\boxed{-3283.83}$

Q6.

i) $u_n = 2n^2 - 1 - 2u_{n-1} \qquad u_1 = 5$

$u_n + 2u_{n-1} = 2n^2 - 1$

Complementary Soln:

Char eqn: $x^2 + 2x = 0$

$x(x+2) = 0$

$x = 0$ or $x + 2 = 0$

$x = -2$

$\Rightarrow \quad u_n = l(0)^n + m(-2)^n$

$= m(-2)^n$

Particular Soln:

$2n^2 - 1$ is quadratic $\Rightarrow u_n = an^2 + bn + c$

$\Rightarrow u_{n-1} = a(n-1)^2 + b(n-1) + c$

$= a(n^2 - 2n + 1) + bn - b + c$

$= an^2 - 2an + a + bn - b + c$

$u_n + 2u_{n-1} = 2n^2 - 1$

$an^2 + bn + c + 2(an^2 - 2an + a + bn - b + c) = 2n^2 - 1$

$an^2 + bn + c + 2an^2 - 4an + 2a + 2bn - 2b + 2c = 2n^2 - 1$

$3an^2 + (3b - 4a)n + 3c + 2a - 2b = 2n^2 + 0n - 1$

$\Rightarrow 3a = 2 \qquad 3b - 4a = 0 \qquad 3c + 2a - 2b = -1$

$a = \frac{2}{3} \qquad 3b = 4a \qquad 3c = -1 - 2(\frac{2}{3}) + 2(\frac{8}{9})$

$3b = 4(\frac{2}{3}) = \frac{8}{3} \qquad 3c = -\frac{5}{9}$

$b = \frac{8}{9} \qquad c = -\frac{5}{27}$

$\Rightarrow$ Total Soln $= m(-2)^n + \frac{2}{3}n^2 + \frac{8}{9}n - \frac{5}{27}$

$u_1 = 5$

$m(-2)^1 + \frac{2}{3}(1)^2 + \frac{8}{9}(1) - \frac{5}{27} = 5$

$-2m = -\frac{14}{9}$

$m = \frac{7}{9} \qquad \Rightarrow u_n = \boxed{\frac{7}{9}(-2)^n + \frac{2}{3}n^2 + \frac{8}{9}n - \frac{5}{27}}$

ii) $u_7 = \frac{7}{9}(-2)^7 + \frac{2}{3}(7)^2 + \frac{8}{9}(7) - \frac{5}{27} = \boxed{-\frac{1643}{27}}$ or $\boxed{-60.85}$

Q7.

i) $T_n = 7T_{n-1} - 12T_{n-2} + 8 - \frac{1}{2}(2)^n \qquad T_1 = 2 \qquad T_2 = 3$

$T_n - 7T_{n-1} + 12T_{n-2} = 8 - \frac{1}{2}(2)^n$

Complementary Soln:

Char eqn: $x^2 - 7x + 12 = 0$

$(x-4)(x-3) = 0$

$x - 4 = 0 \qquad x - 3 = 0$

$x = 4 \qquad x = 3$

$\Rightarrow u_n = l(4)^n + m(3)^n$

Particular Soln:

$8 - \frac{1}{2}(2)^n$ is exponential $\Rightarrow u_n = a(2)^n + b$

$\Rightarrow u_{n-1} = a(2)^{n-1} + b \qquad u_{n-2} = a(2)^{n-2} + b$

$= \frac{a(2)^n}{2} + b \qquad = \frac{a(2)^n}{4} + b$

$T_n - 7T_{n-1} + 12T_{n-2} = 8 - \frac{1}{2}(2)^n$

$a(2)^n + b - 7\left(\frac{a}{2}(2)^n + b\right) + 12\left(\frac{a}{4}(2)^n + b\right) = 8 - \frac{1}{2}(2)^n$

$a(2)^n + b - \frac{7a}{2}(2)^n - 7b + 3a(2)^n + 12b = 8 - \frac{1}{2}(2)^n$

$\frac{a}{2}(2)^n + 6b = 8 - \frac{1}{2}(2)^n$

$\Rightarrow \frac{a}{2} = \frac{-1}{2} \qquad 6b = 8$

$a = -1 \qquad b = \frac{8}{6} = \frac{4}{3}$

$\Rightarrow$ Total Soln: $l(4)^n + m(3)^n - 1(2)^n + \frac{4}{3}$

$T_1 = 2$

$l(4)^1 + m(3)^1 - 1(2)^1 + \frac{4}{3} = 2$

$4l + 3m = \frac{8}{3} \quad : \mathrm{I}$

$T_2 = 3$

$l(4)^2 + m(3)^2 - 1(2)^2 + \frac{4}{3} = 3$

$16l + 9m = \frac{17}{3} \quad : \mathrm{II}$

Solving I & II:

$\mathrm{I} \times 3: \quad 12l + \cancel{9m} = 8$

$\mathrm{II} : (-)16l \overset{(-)}{+} 9m = (-)\frac{17}{3}$

$-4l = \frac{7}{3}$

$l = \frac{-7}{12}$

$\mathrm{I}: 4\left(\frac{-7}{12}\right) + 3m = \frac{8}{3}$

$3m = 5$

$m = \frac{5}{3}$

$\Rightarrow T_n = \boxed{\frac{-7}{12}(4)^n + \frac{5}{3}(3)^n - 2^n + \frac{4}{3}}$

ii) $T_{10} = \frac{-7}{12}(4)^{10} + \frac{5}{3}(3)^{10} - 2^{10} + \frac{4}{3} = \boxed{-514,277}$

Q8.

i) Birth rate = 1.6% Death rate = 1.2%

$\Rightarrow$ net change = 0.4%

Immigration = 40000 Emigration = 25000

$\Rightarrow$ net change = 15000

$\Rightarrow$ Popn in any year will be 1.4% of previous year and then 15000 will be added

$$\Rightarrow \boxed{U_n = 1.004 U_{n-1} + 15000}$$

ii) $\underline{n=1}$

$$U_1 = 1.004 U_0 + 15000$$
$$= 1.004(5M) + 15000$$

$\underline{n=2}$

$$U_2 = 1.004 U_1 + 15000$$
$$= 1.004(1.004(5m) + 15000) + 15000$$
$$= (1.004)^2(5m) + 1.004(15000) + 15000$$

$\underline{n=3}$

$$U_3 = 1.004 U_2 + 15000$$
$$= 1.004((1.004)^2(5m) + 1.004(15000) + 15000) + 15000$$
$$= (1.004)^3(5m) + \underbrace{(1.004)^2(15000) + 1.004(15000) + 15000}_{a = 15000 \quad r = 1.004}$$

$$\Rightarrow U_n = (1.004)^n(5{,}000{,}000) + \frac{15000(1-(1.004)^n)}{1-1.004}$$
$$= (1.004)^n(5{,}000{,}000) - 3750000(1-(1.004)^n)$$
$$= (1.004)^n(5{,}000{,}000 + 3{,}750{,}000) - 3{,}750{,}000$$
$$= \boxed{8{,}750{,}000(1.004)^n - 3{,}750{,}000}$$

iii)
$$U_{30} = 8750000(1.004)^{30} - 3{,}750{,}000$$
$$= \boxed{6{,}113{,}236}$$

Q9. Loan gains 1.2% interest each month

i) $\Rightarrow$ debt is 1.012 of previous month's debt Repays: €300

$\Rightarrow$ $\boxed{U_n = 1.012\,U_{n-1} - 300}$ Initial debt $= U_0 = 10{,}000$

ii) $\underline{n=1}$

$U_1 = 1.012\,U_0 - 300$

$= 1.012(10000) - 300$

$\underline{n=2}$

$U_2 = 1.012\,U_1 - 300$

$= 1.012(1.012(10000) - 300) - 300$

$= (1.012)^2(10000) - 1.012(300) - 300$

$\underline{n=3}$

$U_3 = 1.012\,U_2 - 300$

$= (1.012)^3(10000) - (1.012)^2(300) - 1.012(300) - 300$

$= (1.012)^3(10000) - [\underline{300 + 300(1.012) + 300(1.012)^2}]$

$a = 300 \quad r = 1.012$

$$\Rightarrow U_n = (1.012)^n(10000) - \frac{300(1-(1.012)^n)}{1-1.012}$$

$= (1.012)^n(10000) + 25000(1-(1.012)^n)$

$= (1.012)^n(10000 - 25000) + 25000$

$= \boxed{25000 - 15000(1.012)^n}$

iii) Debt $= 0$ $\Rightarrow$ $25000 - 15000(1.012)^n = 0$

$15000(1.012)^n = 25000$

$(1.012)^n = \frac{25000}{15000} = \frac{5}{3}$

$\log(1.012)^n = \log\left(\frac{5}{3}\right)$

$n\log(1.012) = \log\left(\frac{5}{3}\right)$

$n = \frac{\log(5/3)}{\log(1.012)} = 42.8 = \boxed{43 \text{ months}}$

iv) $U_{42} = 25000 - 15000(1.012)^{42}$

$= €244.438$

$\Rightarrow$ Final payment $= 244.438 \times 1.012$

$= \boxed{€247.37}$

Q10.

i) $D_n = 1.003 D_{n-1} - A$ $\qquad$ $35_{yrs} = 420$ months

ii) $\underline{n=1}$

$D_1 = 1.003 D_0 - A$

$= 1.003(300{,}000) - A$

$\underline{n=2}$

$D_2 = 1.003 D_1 - A$

$= 1.003(1.003(300000) - A) - A$

$= (1.003)^2(300000) - 1.003A - A$

$\underline{n=3}$

$D_3 = 1.003 D_2 - A$

$= 1.003((1.003)^2(300{,}000) - 1.003A - A) - A$

$= (1.003)^3(300{,}000) - \underbrace{[A + 1.003A + (1.003)^2 A]}_{a=A \quad r=1.003}$

$$\Rightarrow D_n = (1.003)^n(300{,}000) - \frac{A(1-(1.003)^n)}{1-1.003}$$

$$= (1.003)^n(300{,}000) + \frac{1000A}{3}(1-(1.003)^n)$$

$\underline{D_n = 0 \quad @ \quad n=420}$

$$(1.003)^{420}(300{,}000) + \frac{1000A}{3}(1-(1.003)^{420}) = 0$$

$$\frac{1000A}{3}(1-(1.003)^{420}) = -300000(1.003)^{420}$$

$$\frac{1000A}{3} = \frac{-300{,}000(1.003)^{420}}{1-(1.003)^{420}}$$

$$\frac{1000A}{3} = 419{,}105.3755$$

$$1000A = 1257316.126$$

$$A = \boxed{€1257.32}$$

Q11. Try with smaller figures i.e. 2 pairs to start

i)

$n = 0$ A B 2 pairs

$n = 1$ A B 2 pairs

$n = 2$ A B (A → C, D; B → E, F) $\}$ $2 + 4 = 6$ pairs

$n = 3$ A B (A → G, H; B → I, J) C, D, E, F $\}$ $2 + 4 + 4 = 10$ pairs

$n = 4$ A B C D E F (A → K, L; B → M, N; C → O, P; D → Q, R; E → S, T; F → U, V), G, H, I, F $\}$ 22 pairs

Sequence: 2, 2, 6, 10, 22 .....

$6 = 2 + 2(2) \qquad 10 = 6 + 2(2) \qquad 22 = 10 + 2(6)$

$$\Rightarrow \boxed{R_n = R_{n-1} + 2R_{n-2}}$$

ii) Char eqn: $x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0$

$x = +2 \qquad x = -1$

Theorem: $R_n = l(2)^n + m(-1)^n$

$\underline{R_0 = 8}$

$l(2)^0 + m(-1)^0 = 8$

$l + m = 8 \quad : \mathrm{I}$

$\underline{R_1 = 8}$

$l(2)^1 + m(-1)^1 = 8$

$2l - m = 8 \quad : \underline{\mathrm{II}}$

Solving I & II:

I: $l + \cancel{m} = 8$

II: $2l - \cancel{m} = 8$

$3l = 16$

$l = 16/3$

I: $\frac{16}{3} + m = 8$

$m = \frac{8}{3}$

$$\Rightarrow R_n = \boxed{\frac{16}{3}(2)^n + \frac{8}{3}(-1)^n}$$

$R_5 = \frac{16}{3}(2)^5 + \frac{8}{3}(-1)^5$

$= \boxed{168}$

Q12.

i) $P_n = \frac{1}{4}(3P_{n-1} + P_{n-2})$

$P_n - \frac{3}{4}P_{n-1} - \frac{1}{4}P_{n-2} = 0$

$4P_n - 3P_{n-1} - P_{n-2} = 0$

Char Eqn:

$4x^2 - 3x - 1 = 0$

$(4x+1)(x-1) = 0$

$4x+1 = 0$ or $x - 1 = 0$

$4x = -1$ $\quad$ $x = 1$

$x = \frac{-1}{4}$

$\Rightarrow$ Theorem: $P_n = l\left(-\frac{1}{4}\right)^n + m(1)^n$

$= l\left(-\frac{1}{4}\right)^n + m$

$P_0 = 70000$ $\qquad$ $P_1 = 95000$

$l\left(-\frac{1}{4}\right)^0 + m = 70000$ $\qquad$ $l\left(-\frac{1}{4}\right)^1 + m = 95000$

$l + m = 70000 \quad : \mathrm{I}$ $\qquad$ $-l + 4m = 380000 \quad : \mathrm{II}$

Solving I & II:

I: $\cancel{l} + m = 70000$

II: $\cancel{-l} + 4m = 380000$

$5m = 450000$

$m = 90000$

I: $l + 90000 = 70000$

$l = -20000$

$\Rightarrow P_n = \boxed{90000 - 20000\left(-\frac{1}{4}\right)^n}$

ii) $2030 \Rightarrow n = 5$

$P_5 = 90000 - 20000\left(-\frac{1}{4}\right)^5$

$= 90019.53$

$= \boxed{90020}$

iii) $\lim_{n\to\infty}\left(90000 - 20000\left(-\frac{1}{4}\right)^n\right) = 90000 - 0 = \boxed{90000}$

Q13.

i) $C_n = C_{n-1} + 0.3C_{n-2} + 5n + 40 \qquad C_1 = 100 \quad C_2 = 175$

$C_3 = 175 + 0.3(100) + 5(3) + 40$

$= \boxed{260}$

$C_4 = 260 + 0.3(175) + 5(4) + 40$

$= 372.5 = \boxed{373}$

ii) Comp Soln:

$C_n - C_{n-1} - 0.3C_{n-2} = 0$

$x^2 - x - 0.3 = 0$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(0.3)}}{2(1)}$$

$$= \frac{5 \pm \sqrt{55}}{10}$$

$$\Rightarrow C_n = l\left(\frac{5+\sqrt{55}}{10}\right)^n + m\left(\frac{5-\sqrt{55}}{10}\right)^n$$

Particular Soln:

$5n + 40$ is linear $\Rightarrow C_n = an + b$

$C_{n-1} = a(n-1) + b = an - a + b$

$C_{n-2} = a(n-2) + b = an - 2a + b$

$C_n - C_{n-1} - 0.3C_{n-2} = 5n + 40$

$an + b - (an - a + b) - 0.3(an - 2a + b) = 5n + 40$

$\cancel{an} + \cancel{b} - \cancel{an} + a - \cancel{b} - 0.3an + 0.6a - 0.3b = 5n + 40$

$-0.3an + 1.6a - 0.3b = 5n + 40$

$-0.3a = 5 \qquad 1.6a - 0.3b = 40$

$a = \frac{-50}{3} \qquad 1.6\left(-\frac{50}{3}\right) - 0.3b = 40$

$-0.3b = \frac{200}{3}$

$b = \frac{-2000}{9}$

$$\Rightarrow \text{Total Soln: } C_n = l\left(\frac{5+\sqrt{55}}{10}\right)^n + m\left(\frac{5-\sqrt{55}}{10}\right)^n - \frac{50n}{3} - \frac{2000}{9}$$

$$\underline{C_1 = 100}$$

$$l\left(\frac{5+\sqrt{55}}{10}\right) + m\left(\frac{5-\sqrt{55}}{10}\right) - \frac{50(1)}{3} - \frac{2000}{9} = 100$$

$$9l(5+\sqrt{55}) + 9m(5-\sqrt{55}) - 1500 - 20000 = 9000 \quad (\times 90)$$

$$9l(5+\sqrt{55}) + 9m(5-\sqrt{55}) = 30500 \quad : \mathrm{I}$$

$$\underline{C_2 = 175}$$

$$l\left(\frac{5+\sqrt{55}}{10}\right)^2 + m\left(\frac{5-\sqrt{55}}{10}\right)^2 - \frac{50(2)}{3} - \frac{2000}{9} = 175$$

$$l\left(\frac{8+\sqrt{55}}{10}\right) + m\left(\frac{8-\sqrt{55}}{10}\right) - \frac{100}{3} - \frac{2000}{9} = 175$$

$$9l(8+\sqrt{55}) + 9m(8-\sqrt{55}) - 3000 - 20000 = 15750$$

$$9l(8+\sqrt{55}) + 9m(8-\sqrt{55}) = 38750 \quad : \mathrm{II}$$

Solving I & II:

$$\mathrm{I}: \quad 111.75l - 21.75m = 30500$$

$$\mathrm{II}: \quad 138.75l + 5.25m = 38750$$

$$\mathrm{I} \times 5.25: \quad 586.6875l - \cancel{114.1875}m = 160125$$

$$\mathrm{II} \times 21.75: \quad 3017.8125l + \cancel{114.1875}m = 842812.5$$

$$3604.5l = 1002937.5$$

$$l = 278.25$$

$$\Rightarrow \mathrm{I}: \quad 111.75(278.25) - 21.75m = 30500$$

$$31094.4375 - 21.75m = 30500$$

$$-21.75m = -594.4375$$

$$m = 27.33$$

$$\Rightarrow C_n = \boxed{278.25\left(\frac{5+\sqrt{55}}{10}\right)^n + 27.33\left(\frac{5-\sqrt{55}}{10}\right)^n - \frac{50n}{3} - \frac{2000}{9}}$$

$$C_{10} = 278.25\left(\frac{5+\sqrt{55}}{10}\right)^{10} + 27.33\left(\frac{5-\sqrt{55}}{10}\right)^{10} - \frac{50(10)}{3} - \frac{2000}{9}$$

$$= 2033.93$$

$$= \boxed{2034}$$

2023 HL Sample Paper Q8

n = 0

=> $P_1 = 1.03P_0 - B$

=> $P_1 = 1.03(8000) - B$

n = 2

=> $P_3 = 1.03P_2 - B$

=> $P_3 = 1.03(1.03^2(8000) - 1.03B - B) - B$

=> $P_3 = 1.03^3(8000) - 1.03^2B - 1.03B - B$

n = 1

=> $P_2 = 1.03P_1 - B$

=> $P_2 = 1.03(1.03(8000) - B) - B$

=> $P_2 = 1.03^2(8000) - 1.03B - B$

Looking at P_1, P_2, P_3 above:

$$P_n = 1.03^n(8000) - [B + B(1.03) + B(1.03)^2 + \cdots]$$

The terms in the square brackets form a Geometric Series with a = B and r = 1.03

=> $P_n = 1.03^n(8000) - [\frac{B(1 - 1.03^n)}{1 - 1.03}]$ (Using the Sn formula from Tables)

=> $P_n = 1.03^n(8000) + \frac{100}{3}[B(1 - 1.03^n)]$

=> $P_n = 1.03^n(8000) + \frac{100}{3}B - \frac{100}{3}B(1.03^n)$

=> $P_n = 1.03^n\left(8000 - \frac{100}{3}B\right) + \frac{100}{3}B$

=> $P_n = 1.03^n\left(\frac{24000 - 100B}{3}\right) + \frac{100}{3}B$

2024 HL Q9

(i)

$M_2 = M_1 + \frac{3M_0}{4} = 245 + \frac{3(200)}{4} = 395$	$M_3 = M_2 + \frac{3M_1}{4} = 395 + \frac{3(245)}{4} = 578.75$

(ii)

$M_{n+2} = M_{n+1} + \frac{3M_n}{4}$

=> $4M_{n+2} - 4M_{n+1} - 3M_n = 0$

=> Characteristic Equation = $4x^2 - 4x - 3 = 0$

$(2x-3)(2x+1) = 0$

=> $2x - 3 = 0$ or $2x + 1 = 0$

=> $x = \frac{3}{2}$ or $x = -\frac{1}{2}$

Theorem: $M_n = l(\frac{3}{2})^n + m(-\frac{1}{2})^n$

$\underline{M_0 = 200}$	$\underline{M_1 = 245}$
$l(\frac{3}{2})^0 + m\left(-\frac{1}{2}\right)^0 = 200$ => $l + m = 200$ => $m = 200 - l$ ……Eqn 1	$l(\frac{3}{2})^1 + m\left(-\frac{1}{2}\right)^1 = 245$ => $3l - m = 490$ Subbing in Eqn 1: => $3l - (200 - l) = 490$ => $4l - 200 = 490$ => $4l = 690$ => $l = \frac{690}{4} = 172.5$ => $m = 200 - 172.5 = 27.5$

=> $M_n = 172.5(\frac{3}{2})^n + 27.5(-\frac{1}{2})^n$

(iii)

$M_0 = 200, M_1 = 245, M_2 = 395, M_3 = 578.75$

$M_4 = 172.5\left(\frac{3}{2}\right)^4 + 27.5\left(-\frac{1}{2}\right)^4 = 875$	$M_5 = 172.5(\frac{3}{2})^5 + 27.5(-\frac{1}{2})^5 = 1309.0625$

Total $= M_0 + M_1 + M_2 + M_3 + M_4 + M_5 = 3602.8125\ kg$

(iv)

$$P_{n+2} = P_{n+1} + \frac{3P_n}{4} - 2^{n+2}$$

=> Equation 1: $4P_{n+2} - 4P_{n+1} - 3P_n = -4.2^{n+2}$ (Multiplying through by 4)

The characteristic equation will be the same as the part (ii), so the complimentary solution will be: $P_n = 172.5(\frac{3}{2})^n + 27.5(-\frac{1}{2})^n$

As the right-hand side is an exponential, Particular solution would be in the form:

$$P_n = k.2^{n+2}$$

=> $P_{n+2} = k.2^{(n+2)+2} = k.2^{n+4}$ and $P_{n+1} = k.2^{(n+2)+1} = k.2^{n+3}$

Subbing into Equation 1 above gives:

$$4k.2^{n+4} - 4k.2^{n+3} - 3k.2^{n+2} = -4.2^{n+2}$$

Breaking up all powers of 2 into the form 2^{n+2} gives:

$$4k.2^{n+2}.2^2 - 4k.2^{n+2}.2^1 - 3k.2^{n+2} = -4.2^{n+2}$$

=> $16k.2^{n+2} - 8k.2^{n+2} - 3k.2^{n+2} = -4.2^{n+2}$

=> $5k.2^{n+2} = -4.2^{n+2}$

=> $5k = -4$

=> $k = -\frac{4}{5}$

=> Total solution will be: $P_n = l(\frac{3}{2})^n + m(-\frac{1}{2})^n - \frac{4}{5}(2)^{n+2}$

$P_0 = 199$	$P_1 = 243$
$l(\frac{3}{2})^0 + m(-\frac{1}{2})^0 - \frac{4}{5}(2)^{0+2} = 199$ => $l + m - \frac{16}{5} = 199$ => $5l + 5m - 16 = 995$ => $5l + 5m = 1011$ => $5m = 1011 - 5l$ .....Eqn 1	$l(\frac{3}{2})^0 + m(-\frac{1}{2})^0 - \frac{4}{5}(2)^{0+2} = 243$ => $\frac{3l}{2} - \frac{m}{2} - \frac{32}{5} = 243$ => $15l - 5m = 2494$ => $15l - (1011 - 5l) = 2494$ (Using Eqn 1) => $20l = 2494 + 1011$ => $l = \frac{3505}{20} = \frac{701}{4} = 175.25$ => $m = \frac{539}{20} = 26.95$

=> $P_n = 175.25(\frac{3}{2})^n + 26.95(-\frac{1}{2})^n - \frac{4}{5}(2)^{n+2}$

(v)

$P_0 = 199, P_1 = 243$

$P_2 = 175.25\left(\frac{3}{2}\right)^2 + 26.95\left(-\frac{1}{2}\right)^2 - \frac{4}{5}(2)^{2+2} = 388.25$	$P_3 = 562.5, P_4 = 837.6875$ $P_5 = 1227.5625$

Total $= P_0 + P_1 + P_2 + P_3 + P_4 + P_5 = 3458\ kg$