- Chapter 10: Differential Equations
- Topic 48: Differential Equations

- Differential Equations are equations that contain a derivative ($\frac{dy}{dx}$) in them.
  E.g. $4x.\frac{dy}{dx} = 5y$
- A 1^{st} order differential equation has a 1^{st} derivative as its highest derivative, whereas a 2^{nd} order differential equation has a 2^{nd} derivative as its highest derivative.
- The differential equations on our course fall into 3 categories:
  - Type 1: 1^{st} Order Differential Equations: General Solutions
  - Type 2: 1^{st} Order Differential Equations with Definite Values
  - Type 3: 2^{nd} Order Separable Differential Equations
- We will look at each of them in turn now.

- a) Type 1: 1^{st} Order Differential Equations: General Solutions

- Example 1: Find the general solution to the differential equation $\frac{dy}{dx} = 5x^2y$.

Solution:

- Firstly, we multiply both sides by dx to eliminate the fractions in the original equation:

$$dy = 5x^2y.dx$$

- We now try and gather all the terms with a 'y' to one side and the 'x' terms to the other side:

$$\frac{1}{y}.dy = 5x^2.dx \qquad \text{(dividing both sides by y)}$$

- We now integrate both sides of the equation using the rules from the last topic.
- It's standard practice to simply put in one constant of integration, and it's normally put on the right hand side of the equation:

$$\int \frac{1}{y}.dy = \int 5x^2.dx \quad \text{(integrating both sides)}$$

$$=> \log_e y = \frac{5x^3}{3} + c \quad \text{(using } \int \frac{1}{x}.dx = \ln x + c \text{ from tables)}$$

- The final step is to get 'y' on its own:

$$\log_e y = \frac{5x^3}{3} + c$$

$$=> y = e^{\frac{5x^3}{3}+c} \qquad \text{(taking } e \text{ of both sides)}$$

Classwork Questions: Pg 184 Ex 10A Qs 2/3/4/6/7/8 and then try Q9

- <u>b) Type 2: 1st Order Differential Equations with Definite Values</u>

- **Example 2:** Find a function $y = f(x)$ such that $\frac{dy}{dx} = 3y^2$ and y = 2 when x = 0.

<u>Solution:</u>

- We start in a similar way to the last type and isolate all the y terms on one side, and the x terms on the other side:

$$\frac{dy}{dx} = 3y^2$$

$$dy = 3y^2.dx \qquad \text{(multiplying both sides by } dx\text{)}$$

$$\frac{1}{y^2}dy = 3.dx \qquad \text{(dividing both sides by } y^2\text{)}$$

- We now integrate both sides again:

$$\int \frac{1}{y^2}dy = \int 3.dx$$

$$\Rightarrow \int y^{-2}dy = \int 3.dx \qquad \text{(rewriting } \frac{1}{y^2} \text{ first as } y^{-2}\text{)}$$

$$\Rightarrow \frac{y^{-1}}{-1} = 3x + c$$

$$\Rightarrow -\frac{1}{y} = 3x + c \qquad \text{(rewriting } y^{-1} \text{ as } \frac{1}{y}\text{)}$$

- We can now apply the other information we were given to evaluate the constant of integration:

When y = 2, x = 0:

$$\Rightarrow -\frac{1}{2} = 3(0) + c$$

$$\Rightarrow c = -\frac{1}{2}$$

- We can now write down our solution with the value of 'c' filled in:

$$-\frac{1}{y} = 3x - \frac{1}{2}$$

- And finally, we rearrange to get 'y' on its own:

$$-\frac{1}{y} = \frac{6x-1}{2} \qquad \text{(tidying up the RHS into a single fraction)}$$

$$\Rightarrow \frac{1}{y} = \frac{-6x+1}{2} \qquad \text{(multiply both sides by -1)}$$

$$\Rightarrow \frac{y}{1} = \frac{2}{-6x+1} \qquad \text{(invert the fractions on both sides)}$$

$$\Rightarrow y = \frac{2}{(1-6x)}$$

- Example 3: Given the differential equation $\frac{dy}{dx} = 2xy^2 + 32x$ and given that y = 4 when x = 0, find the value of y when x = 3. Give your answer correct to 1 decimal place.

Solution:

- As before, we start by eliminating the fractions initially, and then isolating x's and y's:

$dy = (2xy^2 + 32x).dx$

=> $dy = 2x(y^2 + 16).dx$ (factorising out 2x)

=> $\frac{1}{y^2+16}.dy = 2x.dx$ (dividing both sides by $y^2 + 16$)

- Integrating both sides gives:

$\int \frac{1}{y^2+16}.dy = \int 2x.dx$

=> $\int \frac{1}{y^2+4^2}.dy = \int 2x.dx$ (rewriting 16 as 4^2)

=> $\frac{1}{4}\tan^{-1}\frac{y}{4} = \frac{2x^2}{2} + c$ (using $\int \frac{1}{x^2+a^2}.dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$ from tables)

=> $\frac{1}{4}\tan^{-1}\frac{y}{4} = x^2 + c$ (simplifying $\frac{2x^2}{2}$)

- We now fill in the conditions we were given to find 'c':

When y = 4, x = 0:

=> $\frac{1}{4}\tan^{-1}\frac{4}{4} = (0)^2 + c$

=> $c = \frac{1}{4}\tan^{-1}\frac{4}{4} = \frac{1}{4}(\frac{\pi}{4})$ (as $\tan^{-1}\frac{4}{4} = \tan^{-1}1 = 45° = \frac{\pi}{4}rads$)

=> $c = \frac{\pi}{16}$

- We can now write the solution above with 'c' filled in:

=> $\frac{1}{4}\tan^{-1}\frac{y}{4} = x^2 + \frac{\pi}{16}$

- We now have to rearrange to get 'y' on its own:

=> $\tan^{-1}\frac{y}{4} = 4x^2 + \frac{\pi}{4}$ (multiplying across by 4)

=> $\frac{y}{4} = \tan(16x^2 + \pi)$ (taking tan of both sides)

=> $y = 4\tan(16x^2 + \pi)$

- This is the solution to the differential equation but in this example, we are being asked to find the value of y at a particular value of x, so we need to go a few steps further and fill in x:

When x = 3 => $y = 4\tan(16(3)^2 + \pi)$

=> $y = 4\tan(144 + \pi)$

=> $y = 4(-0.5636)$ (making sure calc is in rad mode)

=> $y = -2.3$

Day 1: Classwork Questions: Pg 185 Ex 10B Qs 2/3/5/7/9/11/13/14

Day 2: Classwork Questions: Pg 186 Ex 10B Qs 16/17/19/21/22/23

- c) Type 3: 2nd Order Separable Differential Equations

- Example 4: Solve $\frac{d^2y}{dx^2} = 6\frac{dy}{dx}$ given that y = 1 when $\frac{dy}{dx}$ = 1 and x = 0.

  Solution:

- We start by letting some variable represent $\frac{dy}{dx}$, so in this example we will use 'v':

  Let v = $\frac{dy}{dx}$ => $\frac{dv}{dx} = \frac{d^2y}{dx^2}$

- We now rewrite the first equation we were asked to solve:

  $\frac{d^2y}{dx^2} = 6\frac{dy}{dx}$ becomes $\frac{dv}{dx} = 6v$

- We now proceed as we did in Type 1 and solve this differential equation:

  $\frac{dv}{dx} = 6v$

  => $dv = 6v.dx$ (multiplying both sides by dx)

  => $\frac{1}{v}.dv = 6.dx$ (dividing both sides by v)

  => $\int \frac{1}{v}.dv = \int 6.dx$ (integrating both sides)

  => $\log_e v = 6x + c$

  When $\frac{dy}{dx} = v = 1$, x = 0 (applying the given conditions)

  => $\log_e 1 = 6(0) + c$

  => $c = 0$

  => $\log_e v = 6x + 0$ (filling in 'c' into solution)

  => $\log_e v = 6x$

  => $v = e^{6x}$ (taking e of both sides)

- We now have to work back from this solution and solve for y, using a similar procedure as before:

  $v = e^{6x}$

  => $\frac{dy}{dx} = e^{6x}$

  => $dy = e^{6x}.dx$ (multiplying both sides by dx)

  => $\int dy = \int e^{6x}.dx$ (integrating both sides)

  => $y = \frac{1}{6}e^{6x} + c$

  When $y = 1$, x = 0 (applying the given conditions)

  => $1 = \frac{1}{6}e^{6(0)} + c$

  => $1 = \frac{1}{6}(1) + c$ (as $e^0 = 1$)

  => $c = 1 - \frac{1}{6} = \frac{5}{6}$

  => $y = \frac{1}{6}e^{6x} + \frac{5}{6}\ Or \frac{1}{6}(e^{6x} + 5)$ (filling in the value of c)

Classwork Questions: Pg 186 Ex 10C Qs 1/3/5

Topic 49: Solving Real-Life Problems Using Differential Equations

a) Acceleration:

- Recall from before:

  s = distance/displacement (m)

  v = velocity (m/s)

  a = acceleration (m/s^2)

- As acceleration is the rate of change of velocity:

$$\boxed{Acc = \frac{dv}{dt}}$$

Used to link v and t

- Using the Chain Rule, we can derive an alternative equation for acceleration:

$$Acc = \frac{dv}{dt} = \frac{dv}{ds} \times \frac{ds}{dt}$$

- Ans as $v = \frac{ds}{dt}$, then:

$$\boxed{Acc = v.\frac{dv}{ds}}$$

Used to link v and s

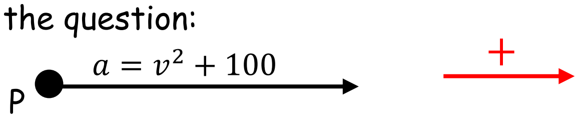
Note: If we want to link s and t, we can use either of the results above i.e. get v in terms of t, or get v in terms of s, and then sub in $\frac{ds}{dt}$ for v.

- **Example 1:** pg 188 Q3

  A particle starts from rest at a point P and moves in a straight line subject to an acceleration equal to $v^2 + 100$, where v is the particle's velocity. Find, correct to two decimal places, the time taken to reach 20 m/s. (Hint: Use radian measure, when calculating $\tan^{-1} 2$.)

  Solution:

- Always sketch a quick diagram of the situation just to check direction of motion and direction of forces in the question:



- It's important to note that we always take the direction of motion as positive.
- In this question, we've been given the acceleration straight away, so we go straight for:

$$a = v^2 + 100$$

$$\Rightarrow \frac{dv}{dt} = v^2 + 100$$

Explain why we pick dv/dt and not v.dv/ds

- Now get all terms with a 'v' to one side and terms with an 't' to the other:

$$\Rightarrow \frac{1}{v^2+100}.dv = dt$$ (dividing both sides by $v^2 + 100$ and multiply by dt)

- We now integrate both sides:

$$\int \frac{1}{v^2+100}.dv = \int 1.dt$$

$$=> \frac{1}{10}\tan^{-1}\frac{v}{10} = t + C$$

- We now use the initial conditions to evaluate the constant of integration C:

  @ t =0, v = 0

  => C = 0

- Subbing C back into our integral above gives:

$$\frac{1}{10}\tan^{-1}\frac{v}{10} = t$$

- We now rearrange to get 'v' in terms of 'v':

  $\tan^{-1}\frac{v}{10} = 10t$ (Multiplying both sides by 10)

  => $\frac{v}{10} = \tan 10t$ (Taking Tan of both sides)

  => $v = 10\tan 10t$ (Multiplying both sides by 10)

- We are interested in where the speed is 20:

  => $20 = 10\tan 10t$

  => $\tan 10t = 2$

  => $10t = \tan^{-1} 2$

  N.B. Calculator in Radian Mode

  => $10t = 1.1\, rads$

  => $t = 0.11 secs$

Help to get started on this one i.e. to get expression for acc

Classwork Questions: Pg 188 Qs 1/2/5/6/8/9

- Now let's look at a slightly trickier example.

- Example 2: Pg 189 Q4

  A particle moves in a straight line and undergoes a retardation of $\frac{v^3}{25}$, where v is the speed.

  i) If the initial speed of the particle is 25 m/s, find its speed when it has travelled a distance of 99 m.

  ii) Find the time for the particle to slow down from 10 m/s to 5 m/s.

  Solution:

  i) In this part, we need to link v and s, so we will start with:

- asd

$$a = -\frac{v^3}{25}$$

Negative as it's deceleration.

$$=> v.\frac{dv}{ds} = -\frac{v^3}{25}$$

- As before, we will get all terms with a 'v' to one side and terms with an 's' to the other:

  => $\frac{1}{v^2}.dv = -\frac{1}{25}.ds$ (divide both sides by v^3 and multiply by ds)

- We now integrate both sides:

$$\int \frac{1}{v^2}.dv = \int -\frac{1}{25}.ds$$

$$=> -\frac{1}{v} = -\frac{1}{25}s + C$$

- We now use the initial conditions to evaluate the constant of integration C:

@ s =0, v = 25

$$=> C = -\frac{1}{25}$$

- Subbing C back into our integral above gives:

$$-\frac{1}{v} = -\frac{1}{25}s - \frac{1}{25}$$

- We now rearrange to get 'v' in terms of 's':

$$\frac{1}{v} = \frac{1}{25}s + \frac{1}{25}$$ (Multiplying both sides by -1)

$$=> \frac{1}{v} = \frac{s+1}{25}$$ (Tidying up the RHS into a single fraction)

$$=> \frac{v}{1} = \frac{25}{s+1}$$ (Inverting both sides)

$$=> v = \frac{25}{s+1}$$

- We are interested in where the distance is 99:

$$=> v = \frac{25}{99+1}$$

$$=> v = 0.25 \text{ m/s}$$

ii) In this part, we need to link v and t, so we will start with:

$$a = -\frac{v^3}{25}$$

$$=> \frac{dv}{dt} = -\frac{v^3}{25}$$

- As before, we will get all terms with a 'v' to one side and terms with an 's' to the other:

$$=> \frac{1}{v^3}.dv = -\frac{1}{25}.dt$$ (divide both sides by v^3 and multiply by dt)

- We now integrate both sides:

$$\int \frac{1}{v^3}.dv = \int -\frac{1}{25}.dt$$

$$=> -\frac{1}{2v^2} = -\frac{1}{25}t + C$$

- We now use the initial conditions to evaluate the constant of integration C:

@ t =0, v = 25

$$=> C = -\frac{1}{1250}$$

- Subbing C back into our integral above gives:

$$-\frac{1}{2v^2} = -\frac{1}{25}t - \frac{1}{1250}$$

- We now rearrange to get 'v' in terms of 't':

$$\frac{1}{2v^2} = \frac{1}{25}t + \frac{1}{1250}$$ (Multiplying both sides by -1)

$$=> \frac{1}{2v^2} = \frac{50t+1}{1250}$$ (Tidying up the RHS into a single fraction)

$$=> 2v^2 = \frac{1250}{50t+1}$$ (Inverting both sides)

$$=> v^2 = \frac{625}{50t+1}$$

$$=> v = \frac{25}{\sqrt{50t+1}}$$

- We are interested in where the time taken to go between 10 m/s and 5 m/s:

If v = 10:	If v = 5:
$10 = \frac{25}{\sqrt{50t+1}}$	$5 = \frac{25}{\sqrt{50t+1}}$
=> $10\sqrt{50t\ +\ 1} = 25$	=> $5\sqrt{50t\ +\ 1} = 25$
=> $t = 0.105\ secs$	=> $t = 0.48\ secs$

- We can now find the time taken to travel between those two speeds:

=> $0.48 - 0.105 = 0.375\ secs$

Day 1: Classwork Questions: Pg 189 Ex 10E Qs 1/3/6 and then try Q7

Day 2: Classwork Questions: Pg 189 Ex 10E Qs 8/9/11/13

- <u>Derivation of Equations of Motion:</u>

- We saw above that acceleration is given by:

$$a = \frac{dv}{dt}$$

$$=> dv = a.dt$$

- We now integrate both sides:

$$\int dv = \int a.dt$$

$$v = at + C$$

$$\underline{@t = 0, v = u}$$

$$=> u = a(0) + C$$

$$=> u = C$$

$$=> v = u + at$$

- We know also that velocity is given by:

$$v = \frac{ds}{dt}$$

$$=> ds = v.dt$$

- Subbing in $v = u + at$ gives:

$$ds = (u + at).dt$$

- Integrating both sides gives:

$$\int ds = \int (u + at).dt$$

$$s = ut + a\left(\frac{t^2}{2}\right) + C$$

$$\underline{@t = 0, s = 0}$$

$$=> 0 = u(0) + a\left(\frac{(0)^2}{2}\right) + C$$

$$=> C = 0$$

$$=> s = ut + \frac{1}{2}at^2$$

- Subbing in $v = u + at$ gives:

$$a = v\frac{dv}{ds}$$

$$=> v.dv = a.ds$$

- Integrating both sides gives:

$$\int v.dv = \int a.ds$$

$$=> \frac{v^2}{2} = as + C$$

$$\underline{@s = 0, v = u}$$

$$=> \frac{(u)^2}{2} = a(0) + C$$

$$=> C = \frac{u^2}{2}$$

$$=> \frac{v^2}{2} = as + \frac{u^2}{2}$$

$$=> v^2 = u^2 + 2as$$

b) Power:

- Recall from before:

$$\boxed{Power = Tv}$$

Where T = Tractive Force, v = velocity

- Example: A car of mass 1200kg starts at rest on a horizontal road. Engine of car is working at constant power of 1500W. Find an expression for the acceleration in terms of its velocity.

  Solution:

  $Power = Fv$

  => $F = \frac{1500}{v} = ma$

  => $\frac{1500}{v} = (1200)a$

  => $a = \frac{5}{4v}$

- You now proceed as we did in the previous questions.

Classwork Questions: Pg 191 Ex 10F Qs 1/3/2/5

c) Populations, Finance and Cooling Problems:

- Let's look at some non-mechanics type problems.

Note on Proportionality:

- If one quantity is proportional to another, it means that one changes in a similar way to the other.
- For example, Voltage in a simple circuit is proportional to the Current flowing through the circuit and we can write that in symbols as $V \propto I$.
- The proportionality relationship means that if the voltage doubles, then the current doubles also, or if the current halves, then the voltage halves also.
- To make a more useful mathematical statement that we can work with, we can write this relationship instead as:

$$V = kI \quad \text{, where k is some constant}$$

- Example: Pg 194 Ex 10G Q5

  Newton's Law of Warming states that 'the rate of warming of a body is proportional to the difference between the temperature of the body and the temperature of its surroundings.'

  i) If θ is the difference between the temperature of a body and the temperature of its surroundings, show that $\frac{d\theta}{dt} = k\theta$.

  ii) A body warms up from 2^oC to 8^oC in 6 minutes in a place where the temperature of the surroundings is a constant 25^oC. Find the value of k to one significant figure.

  iii) What will the temperature of the body be in a further 6 minutes? Give your answer to one decimal place.

  Solution:

  i) θ is the difference between the temperature of a body and the temperature of its surroundings and the rate of warming is given by $\frac{d\theta}{dt}$

  => $\frac{d\theta}{dt} \propto \theta$

  => $\frac{d\theta}{dt} = k\theta$

  ii) Let's rearrange this differential equation as we've done previously:

  $\frac{1}{\theta}.d\theta = k.dt$

- We now integrate both sides:

  $\int \frac{1}{\theta}.d\theta = \int k.dt$

  => $\ln\theta = kt + C$

- We now use the initial conditions to evaluate the constant of integration C, but there are a few things to be cautious of here.
- Firstly, the temperature warms up from 2^oC to 8^oC in 6 minutes, so we take the time as zero at 2^oC when our measurements started i.e. temperature is 2^oC when t = 0.
- Secondly, θ is the difference between the temperature of a body and the temperature of its surroundings, so we have to be careful when subbing in values for θ.
- For example, when the temperature is 2^oC, the value of θ will be 25^oC - 2^oC = 23^oC, so:

  @ t = 0, θ = 23

  => $\ln 23 = k(0) + C$

  => C = $\ln 23$

- Subbing C back into our integral above gives:

  $\ln\theta = kt + \ln 23$

- In this question, we have another set of initial conditions, which allows us to evaluate k:

  @ t = 6, θ = 25 – 8 = 17

  => $\ln 17 = k(6) + \ln 23$

  => $\ln 17 - \ln 23 = k(6)$

  => $6k = \ln\frac{17}{23}$ (Using Law 2 of Logs)

=> $6k = -0.3$

=> $k = -0.05$

iii) We can now rearrange our solution above to tidy it up:

$\ln\theta = kt + \ln 23$

=> $\ln\theta - \ln 23 = (-0.05)t$

=> $\ln\frac{\theta}{23} = -0.05t$

=> $\frac{\theta}{23} = e^{-0.05t}$ (Taking e of both sides)

=> $\theta = 23e^{-0.05t}$

- We are now asked about the temperature in a "further 6 minutes" i.e. when t = 12

=> $\theta = 23e^{-0.05(12)}$

=> $\theta = 12.62°$

- Finally, recall that θ is the difference between the temperature of a body and the temperature of its surroundings, so the temperature of the body has to be:

Temp = $25 - 12.62 = 12.4°C$

Answer at back says 12.6.

Day 1: Classwork Questions: Pg 193/194 Ex 10G Qs 2/3

Day 2: Classwork Questions: Pg 193/194 Ex 10G Qs 1/4/8 (They will need help starting these)

Revision Questions and Test