Revision Sheet 3 - Worked Solutions

Q1. $\frac{4+\sqrt{3}}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}}$

$$= \frac{(4+\sqrt{3})(2-\sqrt{3})}{(2+\sqrt{3})(2-\sqrt{3})}$$

$$= \frac{8+2\sqrt{3}-4\sqrt{3}-3}{4+\cancel{2\sqrt{3}}-\cancel{2\sqrt{3}}-3}$$

$$= \frac{5-2\sqrt{3}}{1}$$

$$= \boxed{5-2\sqrt{3}}$$

Q2. $x^2-(3k+1)x+(2k^2+k+4)=0$

Real roots $\Rightarrow b^2-4ac \geq 0$

$\underline{b^2-4ac}$

$(3k+1)^2-4(1)(2k^2+k+4) \geq 0$

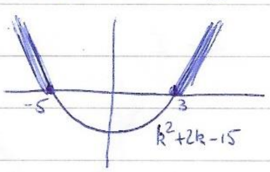
$9k^2+6k+1-8k^2-4k-16 \geq 0$

$k^2+2k-15 \geq 0$

Solve $k^2+2k-15=0$

$(k+5)(k-3)=0$

$k=-5 \quad k=3$



Using graph above

$k^2+2k-15 \geq 0$ when

$\boxed{k \leq -5}$ or $\boxed{k \geq 3}$

Q3. $|2x-1| = 7$

Method 1: (Defn. of modulus)

$2x-1=7$ or $2x-1=-7$

$2x=8 \qquad 2x=-6$

$\boxed{x=4} \qquad \boxed{x=-3}$

Method 2: (Sq both sides)

$(2x-1)^2 = (7)^2$

$4x^2-4x+1=49$

$4x^2-4x-48=0$

$x^2-x-12=0$

$(x+3)(x-4)=0$

$\boxed{x=-3}$ or $\boxed{x=4}$

Q4.

i) $\frac{1}{a}+\frac{1}{b} \geq \frac{2}{a+b}$

Multiply both sides by $ab(a+b)$ and inequality stays as is, as $a,b>0$

$$\cancel{a}b(a+b)\frac{1}{\cancel{a}} + a\cancel{b}(a+b)\frac{1}{\cancel{b}} \geq ab\cancel{(a+b)}\frac{2}{\cancel{a+b}}$$

$ab+b^2+a^2+ab \geq 2ab$

$a^2+b^2+2ab \geq 2ab$

$a^2+b^2+\cancel{2ab}-\cancel{2ab} \geq 0$

$a^2+b^2 \geq 0$

which is true $\forall\, a,b>0$

$\Rightarrow \frac{1}{a}+\frac{1}{b} \geq \frac{2}{a+b}$ Q.E.D.

ii) $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \geq 4$

$1+\frac{a}{b}+\frac{b}{a}+1 \geq 4$

Mult across by ab:

$ab+a^2+b^2+ab \geq 4ab$

$a^2+b^2+2ab-4ab \geq 0$

$a^2+b^2-2ab \geq 0$

$(a-b)(a-b) \geq 0$

$(a-b)^2 \geq 0$ which is true $\forall\, a,b>0$

$\Rightarrow a+b\left(\frac{1}{a}+\frac{1}{b}\right) \geq 4$ Q.E.D.

iii) $a+b \geq 2\sqrt{ab}$

Sq both sides:

$(a+b)^2 \geq (2\sqrt{ab})^2$

$a^2+2ab+b^2 \geq 4ab$

$a^2-2ab+b^2 \geq 0$

$(a-b)^2 \geq 0$

which is true $\forall\ a,b>0$

$\Rightarrow a+b \geq 2\sqrt{ab}$ Q.E.D.

Q5. $(x-a)(x-b)-h^2=0$

$\Rightarrow x^2-ax-bx+ab-h^2=0$

$\Rightarrow x^2-(a+b)x+ab-h^2=0$

Real roots $\Rightarrow b^2-4ac \geq 0$

$\Rightarrow (a+b)^2-4(1)(ab-h^2) \geq 0$

$a^2+2ab+b^2-4ab+4h^2 \geq 0$

$a^2-2ab+b^2+4h^2 \geq 0$

$(a-b)^2+4h^2 \geq 0$

which is always true $\forall\ a,b,h \in R$

Q6. $\dfrac{\sqrt{x+1}-\sqrt{x}}{\sqrt{x+1}+\sqrt{x}} \times \dfrac{\sqrt{x+1}-\sqrt{x}}{\sqrt{x+1}-\sqrt{x}}$

$= \dfrac{x+1+x-\sqrt{x}\sqrt{x+1}-\sqrt{x}\sqrt{x+1}}{\cancel{x}+1-\cancel{x}+\cancel{\sqrt{x}\sqrt{x+1}}-\cancel{\sqrt{x}\sqrt{x+1}}}$

$= \dfrac{2x+1-2\sqrt{x^2+x}}{1}$

$= \boxed{2x+1-2\sqrt{x^2+x}}$

Q7. $4ax^2-4ax+a+c^2=0$

No real roots $\Rightarrow b^2-4ac<0$

$\Rightarrow (-4a)^2-4(4a)(a+c^2) \leq 0$

$\cancel{16a^2}-\cancel{16a^2}-16ac^2 \leq 0$

$-16ac^2 \leq 0$

which is true $\forall\ a \in N$ and $c \in R$

Q8. $\dfrac{x+3}{2x-1} \leq 4$

Multiply both sides by $(2x-1)^2$

$\Rightarrow (x+3)(2x-1) \leq 4(2x-1)^2$

$2x^2+6x-x-3 \leq 4(4x^2-4x+1)$

$2x^2+5x-3 \leq 16x^2-16x+4$

$\Rightarrow 14x^2-21x+7 \geq 0$

$2x^2-3x+1 \geq 0$

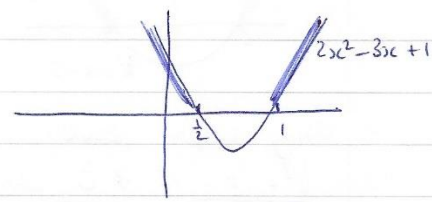
Solve $2x^2-3x+1=0$

$(2x-1)(x-1)=0$

$2x-1=0$ or $x-1=0$

$2x=1$ $\quad$ $x=1$

$x=\frac{1}{2}$



From graph above $2x^2-3x+1 \geq 0$ when $\boxed{x \leq \frac{1}{2}}$ or $\boxed{x \geq 1}$

Q9. i) $p^2 + 4q^2 \geq 4pq$

$p^2 - 4pq + 4q^2 \geq 0$

$(p-2q)(p-2q) \geq 0$

$(p-2q)^2 \geq 0$

which is true $\forall\ p,q \in \mathbb{R}$

$\Rightarrow p^2 + 4q^2 \geq 4pq$

Q.E.D.

ii) $(p+q)^2 \leq 2(p^2+q^2)$

$p^2 + 2pq + q^2 \leq 2p^2 + 2q^2$

$-p^2 + 2pq - q^2 \leq 0$

$p^2 - 2pq + q^2 \geq 0$

$(p-q)(p-q) \geq 0$

$(p-q)^2 \geq 0$

which is true $\forall\ p,q \in \mathbb{R}$

$\Rightarrow (p+q)^2 \leq 2(p^2+q^2)$

Q.E.D.

Q10. $\dfrac{x}{x-1} - \dfrac{3}{2} \geq \dfrac{1}{2(x-1)}$

$\dfrac{2x - 3(x-1)}{2(x-1)} \geq \dfrac{1}{2(x-1)}$

$\dfrac{2x - 3x + 3}{2x-2} \geq \dfrac{1}{2x-2}$

$\dfrac{-x+3}{2x-2} \geq \dfrac{1}{2x-2}$

Multiply both sides by $(2x-2)^2$:

$(2x-2)^2\left(\dfrac{-x+3}{2x-2}\right) \geq (2x-2)^2\left(\dfrac{1}{2x-2}\right)$

$(2x-2)(-x+3) \geq 2x-2$

$-2x^2 + 2x + 6x - 6 - 2x + 2 \geq 0$

$-2x^2 + 6x - 4 \geq 0$

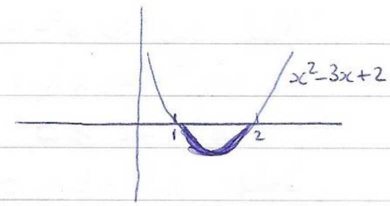
$x^2 - 3x + 2 \leq 0$

Solve $x^2 - 3x + 2 = 0$

$(x-2)(x-1) = 0$

$x - 2 = 0$ or $x - 1 = 0$

$x = 2$ $\quad$ $x = 1$



From graph above $x^2 - 3x + 2 \leq 0$ when $\boxed{1 \leq x \leq 2}$

Q11. $\dfrac{1-\sqrt{3}}{1+\sqrt{3}} \times \dfrac{1-\sqrt{3}}{1-\sqrt{3}}$

$= \dfrac{(1-\sqrt{3})(1-\sqrt{3})}{(1+\sqrt{3})(1-\sqrt{3})}$

$= \dfrac{1 - \sqrt{3} + 3 - \sqrt{3}}{1 + \cancel{\sqrt{3}} - \cancel{\sqrt{3}} - 3}$

$= \dfrac{4 - 2\sqrt{3}}{-2}$

$= -2 + \sqrt{3}$

$= \boxed{\sqrt{3} - 2}$

Q12. $|3x-1| = |5x-7|$

Method 1: (Defn of modulus)

$3x - 1 = 5x - 7$ or $3x - 1 = -5x + 7$

$-2x = -6$ $\quad$ $8x = 8$

$\boxed{x = 3}$ $\quad$ $\boxed{x = 1}$

Method 2: (Sq. both sides)

$(3x-1)^2 = (5x-7)^2$

$9x^2 - 6x + 1 = 25x^2 - 70x + 49$

$16x^2 - 64x + 48 = 0$

$x^2 - 4x + 3 = 0$

$\Rightarrow \boxed{x = 3}$ or $\boxed{x = 1}$

Q13. $kx(1-x) = 1$

$kx - kx^2 = 1$

$-kx^2 + kx - 1 = 0$

$kx^2 - kx + 1 = 0$

Real roots $\Rightarrow b^2 - 4ac \geqslant 0$

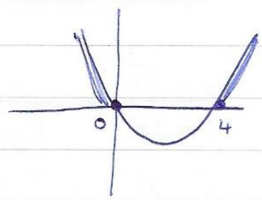
$\Rightarrow (-k)^2 - 4(k)(1) \geqslant 0$

$k^2 - 4k \geqslant 0$

Solve $k^2 - 4k = 0$

$k(k-4) = 0$

$k = 0$ or $k = 4$



From graph $k^2 - 4k \geqslant 0$

when $\boxed{k \leqslant 0}$ or $\boxed{k \geqslant 4}$

Q14. $a^2 - 6a + 9 + b^2 \geqslant 0$

$(a-3)^2 + b^2 \geqslant 0$

which is true $\forall\ a, b \in \mathbb{R}$

$\Rightarrow a^2 - 6a + 9 + b^2 \geqslant 0$

Q.E.D.

Q15. $\sqrt{2x+5} - x = 1$

$\sqrt{2x+5} = 1 + x$

Sq. both sides:

$2x+5 = (1+x)^2$

$2x+5 = x^2 + 2x + 1$

$x^2 - 4 = 0$

$(x+2)(x-2) = 0$

$x = \pm 2$

Checking both eliminates $x = -2$

$\Rightarrow \boxed{x = 2}$

Q16. $\sqrt{6x+4} - 1 = \sqrt{3x+1}$

Sq. both sides:

$\left(\sqrt{6x+4} - 1\right)^2 = \left(\sqrt{3x+1}\right)^2$

$6x+4+1 - 2\sqrt{6x+4} = 3x+1$

$6x+5 - 2\sqrt{6x+4} = 3x+1$

$-2\sqrt{6x+4} = -3x - 4$

Sq. both sides again:

$\left(-2\sqrt{6x+4}\right)^2 = (-3x-4)^2$

$4(6x+4) = 9x^2 + 24x + 16$

$\cancel{24x} + 16 = 9x^2 + \cancel{24x} + 16$

$9x^2 = 0$

$x^2 = 0$

$\Rightarrow \boxed{x = 0}$