Topic 2: Complex Numbers

1) The Basics:

a) Definition:

Notes:

- A complex number is of the form $a + bi$.
- $i = \sqrt{-1}$

$$i^2 = -1$$

b) Adding/Subtracting:

Note:

- Add/Subtract like terms together as we did in Algebra

Example: $z_1 = 3 - 2i$ and $z_2 = 4 + 5i$, evaluate i) $z_1 + z_2$ ii) $z_1 - 2z_2$

i) $z_1 + z_2 = 3 - 2i + 4 + 5i = 7 + 3i$

i) $z_1 - 2z_2 = 3 - 2i - 2(4 + 5i) = 3 - 2i - 8 - 10i = -5 - 12i$

c) Multiplying Complex Numbers:

Notes:

- When multiplying by a constant, we treat as usual
- When multiplying by an 'i' or another complex number, we have to use the definition o.

Example 1: If $z_1 = 2 - 5i$, evaluate $3z_1$.

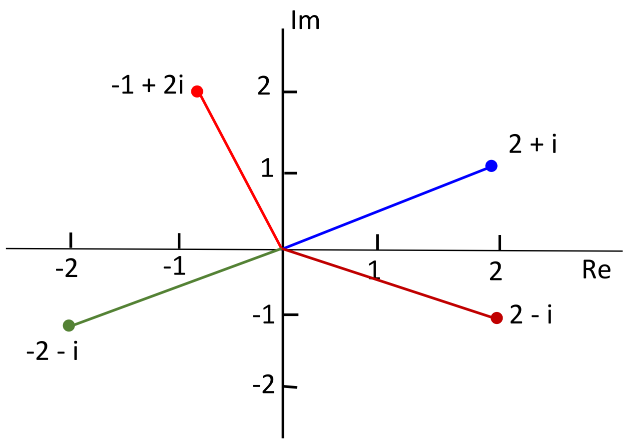
$3z_1 = 3(2 - 5i)$
$= 6 - 15i$

Example 2: If $z_1 = 3 + 4i$ and $z_2 = -2 - 5i$, evaluate $z_1.z_2$.

$z_1.z_2 = (3 + 4i)(-2 - 5i)$
$= 3(-2 - 5i) + 4i(-2 - 5i)$
$= -6 - 15i - 8i - 20i^2$
$= -6 - 15i - 8i - 20(-1)$
$= -6 - 15i - 8i + 20$
$= 14 - 23i$

2) Argand Diagram/Modulus:

a) Argand Diagram:



b) Modulus:

Notes:

- Modulus is the distance from the origin (0, 0) to the complex number.

If $z = a + bi$
$|z| = \sqrt{a^2 + b^2}$

- When finding |z|, ensure z is in the form a + bi first.

Example: If $z_1 = 3 + 4i$ and $z_2 = -2 - 5i$, evaluate i) $|z_2|$ and ii) $|3z_2 - 2z_1|$

i) $|z_1| = \sqrt{(-2)^2 + (-5)^2}$ $= \sqrt{29}$	ii) Tidy up the expression into the form a + bi first: $3z_2 - 2z_1 = 3(-2 - 5i) - 2(3 + 4i)$ $= -6 - 15i - 6 - 8i = -12 - 23i$ => $|3z_2 - 2z_1| = |-12 - 23i| =$ $\sqrt{(-12)^2 + (-23)^2} = \sqrt{673} = 25.9$

3) Conjugates/Division of Complex Numbers:

a) Conjugate:

If $z = a + bi$
=> Conjugate = $\bar{z} = a - bi$

Examples:

i) If $z_1 = 3 + 4i$ => $\bar{z_1} = 3 - 4i$

ii) If $z_2 = -2 - 5i$ => $\bar{z_2} = -2 + 5i$

iii) If $z_3 = 3i - 1$ => $\bar{z_3} = -3i - 1$

b) Division of Complex Numbers:

Multiply above and below by the conjugate of the denominator.

Example: Evaluate $\frac{3+5i}{2-3i}$.

$$\frac{3+5i}{2-3i} \times \frac{2+3i}{2+3i}$$

$$= \frac{(3\ +\ 5i)(2\ +\ 3i)}{(2\ -\ 3i)(2 + 3i)} = \frac{6 + 9i\ +\ 10i + 15i^2}{4 + 6i -\ 6i - 9i^2}$$

$$= \frac{6 + 9i + 10i - 15}{4 + 6i - 6i + 9} = \frac{-9 + 19i}{13} = -\frac{9}{13} + \frac{19}{13}i$$

4) Solving Equations involving Complex Numbers:

a) Quadratic Equations with Complex Roots:

Notes:

- To solve an equation of the form $az^2 + bz + c = 0$, we can use the '-b Formula' from Algebra Topic – Section 3d

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example: Solve the equation $z^2 - 2z + 10 = 0$.

=> a = 1, b = -2 and c = +10

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$z = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(10)}}{2(1)}$$

$$z = \frac{2 \pm \sqrt{-36}}{2}$$ (tidying up under the $\sqrt{\ }$)

$$z = \frac{2 \pm \sqrt{-1}\sqrt{36}}{2}$$ (as $\sqrt{ab} = \sqrt{a}\sqrt{b}$)

$$z = \frac{2 \pm 6i}{2}$$ (using definition of i from Section 1a)

$$z = 1 \pm 3i$$

b) Forming Quadratic Equation when given Roots:

Notes:

- To find the quadratic equation from given complex roots, we use the formula below (which needs to be known).
- If the quadratic equation has **real roots**, then one root will always be a conjugate of another. (**Conjugate Root Theorem**)

$$z^2 - (Sum\ of\ roots)z + (Product\ Roots) = 0$$

Example: Form a quadratic equation where one root is $3 - 4i$, giving your answer in the form $z^2 + bz + c = 0$, where $b, c \in Z$.

- Using the conjugate root theorem, if one root is $3 - 4i$ then the other root has to be $3 + 4i$

$$z^2 - (Sum\ of\ roots)z + (Product\ Roots) = 0$$

=> $z^2 - (3 - 4i + 3 + 4i)z + (3 - 4i)(3 + 4i) = 0$

=> $z^2 - 6z + 25 = 0$

c) Solving Cubic Equations with Real Coefficients:

Notes:

- To show a root is a root of an equation, fill in the root for z and see if LHS = RHS.
- As long as coefficients are real, conjugate root theorem also applies to help find roots.
- If two roots are known, form quadratic equation using method from part (b) and then use long division to find the third factor/root.

Example: $f(z) = 2z^3 - 9z^2 + 30z - 13$ is a cubic polynomial. Solve f(z) = 0, if $2 + 3i$ is a root.

$$z^2 - (Sum\ of\ roots)z + (Product\ Roots)$$

=> $z^2 - (2 + 3i + 2 - 3i)z + (2 + 3i)(2 - 3i)$

=> $z^2 - 4z + 13$ is a factor of the cubic equation

$$\begin{array}{r|l} & 2z - 1 \\ z^2 - 4z + 13 & 2z^3 - 9z^2 + 30z - 13 \\ & \underline{(-)\,2z^3 \overset{(+)}{-} 8z^2} \\ & \quad -z^2 + 30z - 13 \\ & \quad \underline{\overset{(+)}{-}z^2 \overset{(-)}{+} 30z \overset{(+)}{-} 13} \\ & \qquad 0 \end{array}$$

$2z^3 - 9z^2 + 30z - 13 = 0$

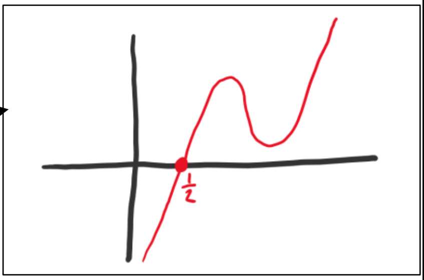
$(2z - 1)(z^2 - 4z + 13) = 0$

=> $2z - 1 = 0 \quad or \quad z^2 - 4z + 13 = 0$

=> $2z = 1 \quad or \quad z = 2 \pm 3i$

=> $z = \frac{1}{2}$

Graph of function with 1 real root and 2 complex roots



d) Solving Equations with Complex Coefficients:

- **Remember:** the conjugate root theorem doesn't apply

Example: $1 + 2i$ is a root of the equation $z^2 - (2 - i)z + 7 - i = 0$. Find its other root.

Method 1:

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

=> $z = \frac{(2 - i) \pm \sqrt{(-(2-i))^2 - 4(1)(7-i)}}{2(1)}$

=> $z = \frac{2 - i \pm \sqrt{-25}}{2}$

=> $z = \frac{2 - i \pm 5i}{2}$

=> $z = \frac{2 + 4i}{2}$ or $z = \frac{2 - 6i}{2}$

=> $z = 1 + 2i$ or $z = 1 - 3i$

Method 2:

- Let unknown root = $a + bi$.
- We know that any quadratic equation will be given by:

$$z^2 - (Sum\ of\ roots)z + (Product\ Roots) = 0$$

- So, the sum of the roots must be equal to $2 - i$:

Sum of Roots = $1 + 2i + a + bi = (1 + a) + (b + 2)i$

- We can now match up the real and imaginary parts to form some equations:

$1 + a = 2 \quad and \quad b + 2 = -1$

=> $a = 1 \quad and \quad b = -3$

=> $z = 1 - 3i$

- Note that we could also have used the product of the roots being equal to $(7 - i)$ to find a and b.

5) Polar Form/DeMoivre's Theorem:

a) Writing Complex Numbers in Polar Form:

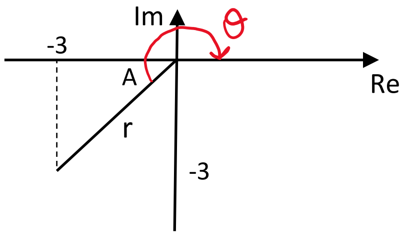
Steps:

1) Draw a rough sketch of the complex number.
2) Calculate the value of r.
3) Calculate the value of θ, paying attention to the diagram.
4) Write down the polar form.

$$r(\cos\theta + i\sin\theta), where\ r = |x + iy|$$

Example: Write $-3 - 3i$ in polar form.

Step 1: Rough sketch of the complex number.



Step 2: Calculate the value of r.

$r^2 = (3)^2 + (3)^2$

=> $r = \sqrt{18} = 3\sqrt{2}$

Step 3: Calculate the value of A and then θ.

$\tan A = \frac{OPP}{ADJ} = \frac{3}{3}$ => $A = 45°\ or\ \frac{\pi}{4} radians$

=> $\theta = 45° + 180°\ or\ \frac{\pi}{4} + \pi$

=> $\theta = 225°\ or\ \frac{5\pi}{4} radians$

Step 4: Write down the polar form.

=> z = $3\sqrt{2}(\cos 225 + i\sin 225)$ or z = $3\sqrt{2}(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4})$

b) DeMoivre's Theorem:

- If $z = r(\cos\theta + i\sin\theta)$

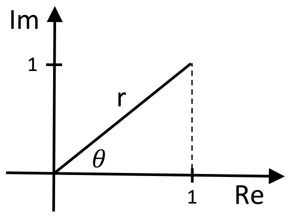
=>

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

c) Using DeMoivre to evaluate large powers:

Example: Write $(1+i)^8$ in the form $a + bi, where\ a, b \in R$.

Step 1: Rough sketch of the complex number $1 + i$:



Step 2: Calculate the value of r.

$r^2 = (1)^2 + (1)^2$ => $r = \sqrt{2}$ (Pyt Thm)

Step 3: Calculate the value of θ.

$\tan\theta = \frac{OPP}{ADJ} = \frac{1}{1}$ => $\theta = \tan^{-1}\frac{1}{1} = 45°\ or\ \frac{\pi}{4} radians$

Step 4: Write down the polar form.

=> z = $\sqrt{2}(\cos 45 + i\sin 45)$ or z = $\sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$

Step 5: Apply DeMoivre's Theorem..

$(1+i)^8 = (\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right))^8$ (filling in the polar form of $1 + i$)

$= (\sqrt{2})^8)\left(\cos\frac{8\pi}{4} + i\sin\frac{8\pi}{4}\right)$ (as $z^n = r^n(\cos n\theta + i\sin n\theta)$)

$= 16(\cos 2\pi + i\sin 2\pi) = 16(1 + 0i) = 16$

d) Using DeMoivre to solve complex equations:

Steps:

1) Rearrange/manipulate to get z on its own.
2) Convert the complex number in brackets to polar form.
3) Rewrite polar form in general polar form.
4) Apply DeMoivre's Theorem.
5) Fill in values of n = 0, 1, 2.... to get the correct number of roots.
6) Check with a quick sketch that all solutions spread evenly around 360° i.e. they should have equal angles between them

Example: Use DeMoivre's Theorem to solve $z^2 = 1 + \sqrt{3}i$.

Step 1: Rearrange/manipulate to get z on its own.

$z^2 = 1 + \sqrt{3}i$ => $z = (1 + \sqrt{3}i)^{1/2}$ (taking the square root of both sides and $\sqrt{x} = x^{1/2}$)

Step 2: Convert the complex number in brackets to polar form.

$$1 + \sqrt{3}i = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$$

Step 3: Rewrite polar form in general polar form.

$$2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$
$$= 2(\cos(\frac{\pi}{3} + 2n\pi) + i\sin(\frac{\pi}{3} + 2n\pi))$$

Step 5: Fill in values of n = 0, 1, 2.... to get all the roots.

When n = 0 => $z = 2^{1/2}\left(\cos\left(\frac{\pi + 6(0)\pi}{6}\right) + i\sin\left(\frac{\pi + 6(0)\pi}{6}\right)\right)$

$= \sqrt{2}(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = \sqrt{2}(\frac{\sqrt{3}}{2} + i\frac{1}{2}) = \frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}i$

When n = 1 => $z = 2^{1/2}\left(\cos\left(\frac{\pi + 6(1)\pi}{6}\right) + i\sin\left(\frac{\pi + 6(1)\pi}{6}\right)\right)$

$= \sqrt{2}(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}) = \sqrt{2}(-\frac{\sqrt{3}}{2} - i\frac{1}{2}) = -\frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}i$

e) Using DeMoivre to prove trigonometric identities:

Example: Use DeMoivre's Theorem to prove the identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

- In this example, we are interested in $\cos 3\theta$, so we will use the expansion of $(\cos\theta + i\sin\theta)^3$:

  $(x + y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$

=>$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3\cos^2\theta\,(i\sin\theta) + 3\cos\theta\,(i\sin\theta)^2 + (i\sin\theta)^3$

=>$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3\cos^2\theta\sin\theta\, i + 3\cos\theta\, i^2\sin^2\theta + i^3\sin^3\theta$

- We know $i^2 = -1$ and $i^3 = -i$

=>$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3\cos^2\theta\sin\theta\, i - 3\cos\theta\sin^2\theta - \sin^3\theta\, i$

- And then we group the real and imaginary parts together on the right-hand side:

  $(\cos\theta + i\sin\theta)^3 = (\cos^3\theta - 3\cos\theta\sin^2\theta) + (3\cos^2\theta\sin\theta - \sin^3\theta)i$

- We now use DeMoivre's Theorem on the expression on the left-hand side above:

=>$\cos 3\theta + i\sin 3\theta = (\cos^3\theta - 3\cos\theta\sin^2\theta) + (3\cos^2\theta\sin\theta - \sin^3\theta)i$

- We now can equate the real/imaginary parts on both sides, depending on what we are looking for i.e. if we want $\cos 3\theta$, we equate the real parts and if we want $\sin 3\theta$ we equate the imaginary parts

  Real = Real

  => $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

- We usually need to use some trig identities, which we looked at last year, to finish from here:

  $\cos 3\theta = \cos^3\theta - 3\cos\theta\,(1 - \cos^2\theta)$ (as $\sin^2\theta + \cos^2\theta = 1$)

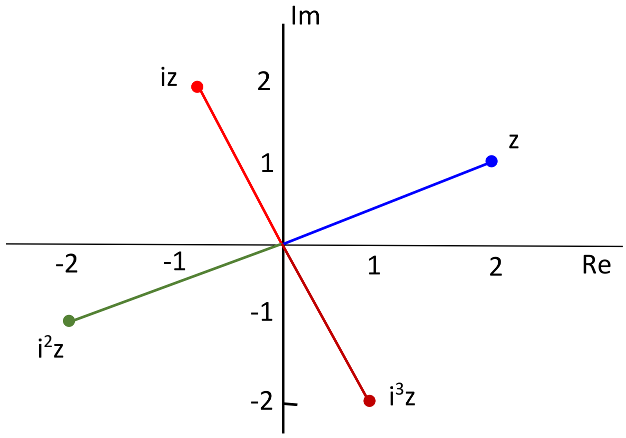
  => $\cos 3\theta = \cos^3\theta - 3\cos\theta + 3\cos^3\theta$

  => $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ Q.E.D.

6) Transformations:

a) Multiplication by i, i^2, i^3 etc:

Effect: Rotation by 90° CCW each time

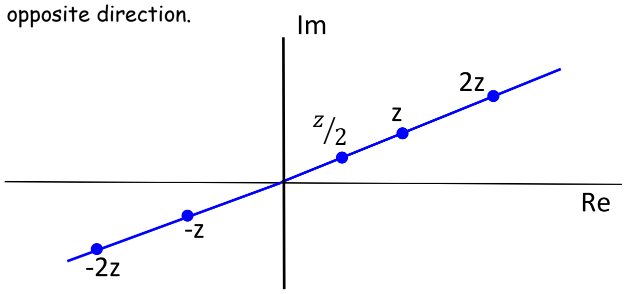


b) Multiplication by a real number:

Effect: Dilation by the value of the real number

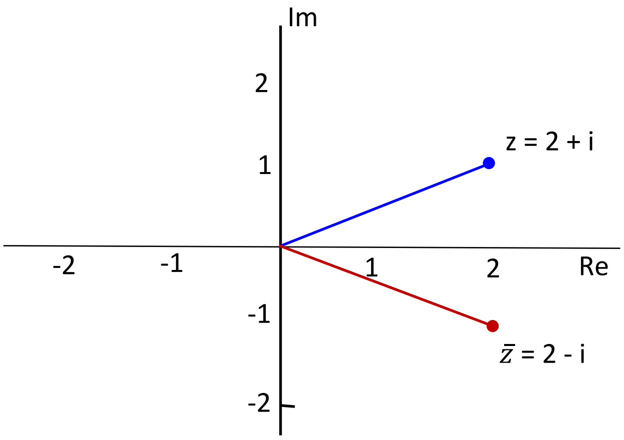
Note 1: If real number is a fraction less than 1, then transformation will be smaller than original.

Note 2: If real number is negative, then transformation will be in opposite direction.



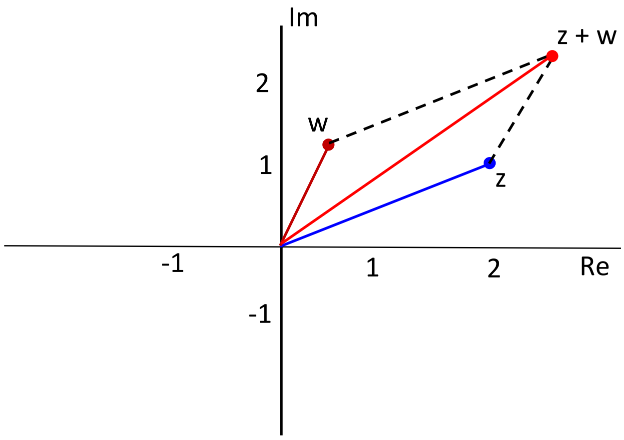
c) Conjugates:

Effect: Conjugate of a complex number will be its image under axial symmetry in the X-AXIS.



d) Addition:

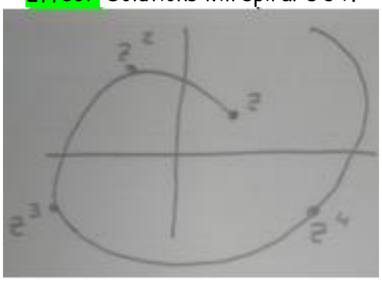
Effect: When adding two complex numbers, we can plot the two of them and complete the parallelogram to find the answer.



e) Multiplying a Complex Number by itself:

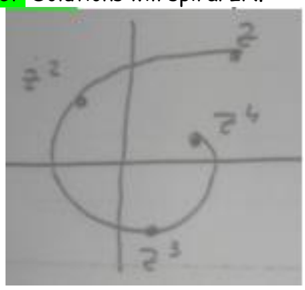
If $|z| > 1$.

Effect: Solutions will spiral OUT.



If $|z| < 1$.

Effect: Solutions will spiral IN.



If $|z| = 1$.

Effect: Solutions will rotate on the same radius.