## Topic 7: Difference Equations

### 1) Arithmetic Sequences/Series:

#### a) Linear(Arithmetic) Sequences:

**Notes:**

- A list of numbers where the **difference** between **each term** is the **same** every time. E.g. 3, 8, 13, 18, ……
- The General Term for an Arithmetic sequence is:

$$T_n = a + (n-1)d$$

See Tables pg 22

where 'a' is the first term and 'd' is the common difference between the terms.

#### b) Arithmetic Series:

**Notes:**

- If we add the terms of an arithmetic sequence together, then we get an arithmetic **series**.
- We need to be able to find the sum of the first n terms of such a series, which we can find using:

$$S_n = \frac{n}{2}\{2a + (n-1)d\}$$

See Tables pg 22

where 'a' is the first term and 'd' is the common difference between the terms of the series.

### 2) Quadratic Sequences:

**Notes:**

- A sequence where the **second difference** is the **same** every time. E.g. 4, 7, 12, 19, 28……. (see below)


**Steps to find General Term:**

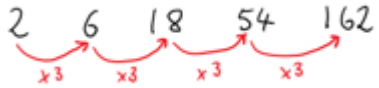
1. Let General Term = $T_n = an^2 + bn + c$
2. Find 2nd difference and let = 2a….solve for a.
3. Use any 2 terms to form two equations in b and c.
4. Solve both equations to find b and c.

### 3) Geometric Sequences/Series:

#### a) Geometric Sequences:

**Notes:**

- A sequence where each term is found by **multiplying** the previous term by the same number every time.




- The General Term for a Geometric sequence is:

$$T_n = a.r^{n-1}$$

See Tables pg 22

where 'a' is the first term and 'd' is the common difference.

#### b) Geometric Series:

**Notes:**

- If we add the terms of an geometric sequence together, then we get a geometric **series**.
- We need to be able to find the sum of the first n terms of such a series, which we can find using:

$$S_n = \frac{a(1-r^n)}{1-r}$$

See Tables pg 22

where 'a' is the first term and 'r' is the common ratio

### 4) Infinite Series/Limits of a Sequence:

#### a) Infinite Series:

**Notes:**

- Series where the terms of an infinite Geometric sequence are added up.

$$S_\infty = \frac{a}{1-r}$$

See Tables pg 22

(for an infinite Geometric Series, where $|r| < 1$)

#### c) Recurrence Relations:

**Notes:**

- A sequence which is defined showing how any term is connected to the previous term.

#### b) Limit of a Sequence:

- Sometimes a sequence can be approaching a particular number e.g. 1, ½, ¼…… is a sequence that approaches 0.
- If a sequence approaches a certain number L, as the number of terms increases, then we say:

$$\lim_{n\to\infty} T_n = L$$

- Another very useful property of limits is:

$$\lim_{n\to\infty} \frac{1}{n^p} = 0$$

### 5) Solving Difference Equations:

#### a) 1st Order Difference Equations:

**Notes:**

- Equations with one term and a previous term e.g. $T_{n+1} = T_n + 5$

#### d) 2nd Order Inhomogeneous Difference Equations:

Steps for solving:

- For interest repayments, debt owing at the end of the term of the loan = 0

Steps for solving:

1. Let n = 1, 2, 3 and write out the first few terms: $T_1, T_2, T_3$
2. Leave in expanded form, to make it easier to spot the pattern
3. Watch for Geometric Series, that we can find the sum of, using the formula in 3(b) above

**b) 2nd Order Difference Equations:**

**Notes:**

- Equations with one term and the previous two terms e.g. $2u_{n+2} - 11u_{n+1} + 5u_n = 0$
- Two types:
  - Homogeneous: $2u_{n+2} - 11u_{n+1} + 5u_n = 0$
  - Inhomogeneous: $2u_{n+2} - 11u_{n+1} + 5u_n = 7n - 14$

**c) 2nd Order Homogeneous Difference Equations:**

Steps for solving:

1. Form characteristic equation i.e. $2x^2 - 11x + 5 = 0$ and solve.
2. Use the theorems below:

*If roots are* ***different***: $u_n = la^n + mb^n$

*If roots are* ***same***: $u_n = la^n + mnb^n$

3. Use terms given to evaluate $l$ and $m$.
4. Write down solution.

1. Move all the terms that are related to each other to the LHS and leave other terms on the RHS. E.g. $2u_{n+2} - 11u_{n+1} + 5u_n = 7n$
2. Solve characteristic equation as in 5(b) to find **complimentary solution**.

E.g. for $2u_{n+2} - 11u_{n+1} + 5u_n = 7n - 14$ the characteristic equation is $2x^2 - 11x + 5 = 0$

2a. To find **particular solution**:

$If\ RHS = Constant \Rightarrow u_n = a$

$If\ RHS = an \Rightarrow u_n = an + b$

$If\ RHS = an^2 \Rightarrow u_n = an^2 + bn + c$

$If\ RHS = a^n \Rightarrow u_n = k.a^n + b$

2b. Get expressions for $u_{n+1}$ and $u_{n+2}$ and sub in to original equation to find the values of $a$ and $b$.

4. Combine particular solution and complimentary solution to get the **total solution**.
5. Use terms given to evaluate $l$ and $m$.
6. Write down solution.

**6) General Tips for the Exam:**

- Take care not to solve the simultaneous equations to find $l$ and $m$ until you have combined the particular and complimentary solutions.
- Make sure you know the two theorem results from 5(c).