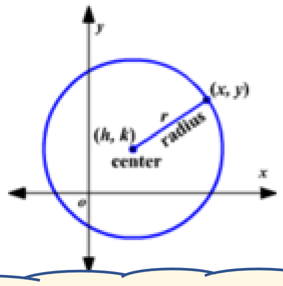
Topic 15: Coordinate Geometry of the Circle

1) The Basics:

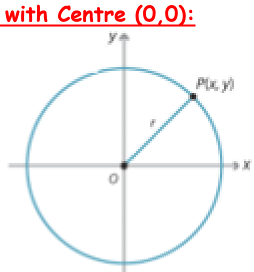
a) Circle with Centre other than (0,0)



Tables pg 19

Equation: $(x-h)^2 + (y-k)^2 = r^2$

b) Circle with Centre (0,0):



Equation: $x^2 + y^2 = r^2$

Not in Tables but can find by subbing in (0,0) for (h,k) in other formula on the left.

2) General Equation of a Circle:

Notes:

- A circle with centre $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$ has the general equation:

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Example:

i) Write down the equation of the circle in the form $x^2 + y^2 + 2gx + 2fy + c = 0$ with centre (-5, 2) and radius 3.

ii) Find the centre and radius of the circle with equation $x^2 + y^2 - 10x + 11y + 14 = 0$.

i) $r = \sqrt{g^2 + f^2 - c}$

=> $3 = \sqrt{(5)^2 + (-2)^2 - c}$

=> $9 = 25 + 4 - c$ (squaring both sides)

=> $c = 20$

=> Eqn: $x^2 + y^2 + 2(5)x + 2(-2)y + 20 = 0$

=> $x^2 + y^2 + 10x - 4y + 20 = 0$

ii)The equation $x^2 + y^2 - 10x + 11y + 14 = 0$ is in the form $x^2 + y^2 + 2gx + 2fy + c = 0$ so comparing both gives:

$2g = -10$ $\quad\quad$ $2f = 11$

=> $g = -5$ $\quad\quad$ $f = \frac{11}{2}$

- Once we know g and f, we can work out the radius:

$r = \sqrt{g^2 + f^2 - c}$

$r = \sqrt{(-5)^2 + (\frac{11}{2})^2 - 14}$

$r = \sqrt{25 + \frac{121}{4} - 14}$

$r = \sqrt{\frac{165}{4}}$

=> Centre = $(-g, -f) = (5, -\frac{11}{2})$ and Radius = $\sqrt{\frac{165}{4}}$

3) Points Inside, On or Outside a Circle:

Method 1:

Steps:

1. Write down the radius and centre of the circle.
2. Calculate distance from the point to the centre.
3. Compare distance to radius:
   - If Distance < Radius => Point is Inside
   - If Distance > Radius => Point is Outside
   - If Distance = Radius => Point is On Circle

Method 2:

Steps:

1. Fill in point into equation of the circle.
2. Compare left hand side to right hand side.
   - If LHS < RHS => Point is Inside
   - If LHS > RHS => Point is Outside
   - If LHS = RHS => Point is On Circle

Example: Is the point (6, -2) in, on or outside the circle $(x-2)^2 + (y+3)^2 = 25$

Method 1:

R = $\sqrt{25}$ = 5 Centre = (2, -3)

Dist from (2,-3) to (6,-2):

$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$\sqrt{(6-2)^2 + (-2+3)^2}$

$\sqrt{17} = 4.12$

$4.12 < 5$

=> INSIDE circle

Method 2:

$(x-2)^2 + (y+3)^2 = 25$

$(6-2)^2 + (-2+3)^2 = 25$

$(4)^2 + (1)^2 = 25$

$17 < 25$

=> INSIDE circle

4) Intersection of a Line and a Circle:

- Need to be able to find the points of intersection of a line and a circle.
- See Algebra Topic Section 8b

5) Tangents/Touching Circles:

a) Tangents:

Notes:

- Tangent: touches a circle at one point and is perpendicular to the radius at the point of contact.

Steps:

1. Find the centre and radius of the circle from the equation.
2. Let the slope of the tangent be 'm'.
3. Find the general equation of the tangent in the form ax + by + c = 0.
4. Find the perpendicular distance from the tangent to the centre, in terms of m.
5. Let the expression from step 4 equal to the radius and solve for m.
6. Fill in the values of 'm' into the equation from step 3 and tidy up.

Example: Find the equations of two tangents from the point (5, -3) to the circle $x^2 + y^2 - 4x + 8y + 12 = 0$.

Step 1: Find the centre and radius from the equation:

Centre = (2, -4) Radius = $\sqrt{(-2)^2 + (4)^2 - 12} = \sqrt{8}$

Step 2: Let the slope of the tangent be 'm'.

Step 3: General equation of the tangent in the form ax + by + c = 0. - (5, -3) is on the tangent so it's equation will be:

$y - y_1 = m(x - x_1)$

=> $y - (-3) = m(x - 5)$

=> $y + 3 = m(x - 5)$

=> $y + 3 = mx - 5m$

=> $mx - y - 3 - 5m = 0$

Step 4: Perpendicular distance from the tangent to the centre (2, -4), in terms of m:

d = $\frac{|m(2) - 1(-4) - 3 - 5m|}{\sqrt{(m)^2 + (-1)^2}}$

=> d = $\frac{|2m + 4 - 3 - 5m|}{\sqrt{m^2 + 1}}$

=> d = $\frac{|-3m + 1|}{\sqrt{m^2 + 1}} = \sqrt{8}$ (Step 5)

=> $8 = \frac{9m^2 - 6m + 1}{m^2+1}$ (squaring both sides)

=> $8(m^2 + 1) = 9m^2 - 6m + 1$

=> $8m^2 + 8 = 9m^2 - 6m + 1$

=> $m^2 - 6m - 7 = 0$

=> $(m - 7)(m + 1) = 0$

=> $m = 7 \quad or \quad m = -1$

Step 6: Filling in our two values of m gives:

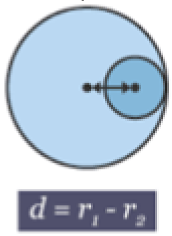
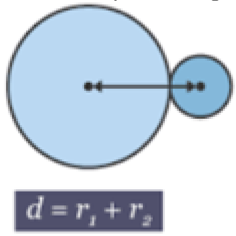
If m = 7:	If m = -1:
$mx - y - 3 - 5m = 0$	$mx - y - 3 - 5m = 0$
$7x - y - 3 - 5(7) = 0$	$-x - y - 3 - 5(-1) = 0$
$7x - y - 38 = 0$	$x + y - 2 = 0$

b) Touching Circles:

Notes:

- **As**

If r_1 = the radius of the larger circle and r_2 = the radius of the smaller circle then the distance between their centres 'd' is given by

Internally Touching	Externally Touching
	

Example: Prove that the circles $s: x^2 + y^2 - 16y + 32 = 0$ and $k: x^2 + y^2 - 18x + 2y + 32 = 0$ touch externally and find their point of contact.

- Firstly, we will write down the centre and radii of both circles:
  s: Centre = (0, 8), Radius = $\sqrt{32} = 4\sqrt{2}$ k: Centre = (9, -1), Radius = $\sqrt{50} = 5\sqrt{2}$
- Now we will calculate the distance between their centres using the distance formula:

  $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

  => $d = \sqrt{(9 - 0)^2 + (-1 - 8)^2}$

  => $d = \sqrt{81 + 81}$

  => $d = \sqrt{162} = 9\sqrt{2}$
- So, the circles must touch externally as $d = 9\sqrt{2} = 4\sqrt{2} + 5\sqrt{2}$.
- To find the point of contact, we can use another formula we learned about in TY for finding a point that divides a line segment up in a particular ratio (which in this case is $4\sqrt{2}:5\sqrt{2}$ or $4:5$

  => P = $\left(\frac{bx_1 + ax_2}{b + a}, \frac{by_1 + ay_2}{b + a}\right)$

  => P = $\left(\frac{5(0) + 4(9)}{5 + 4}, \frac{5(8) + 4(-1)}{5 + 4}\right)$

  => P = $\left(\frac{36}{9}, \frac{36}{9}\right)$ = (4, 4)

6) Problems in g, f and c:

Tips for solving circle problems:

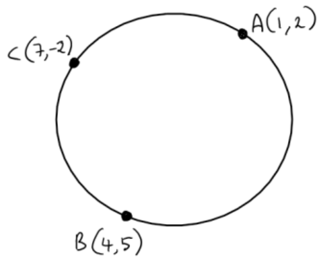
1) Draw a diagram.
2) Find 3 equations because we have three unknowns i.e. g, f and c

3 common types of questions:

a) 3 points on a circle
b) 2 points on a circle and the centre on a line
c) 1 point on a circle, the centre on a line and radius length given

Example 1: A circle passes through the points (7, -2), (1, 2) and (4, 5). Find the equation of the circle.

- Firstly, let's draw a rough sketch of the situation here:



- In this example, we have been given three points on the circle, so we can fill these into our general equation $x^2 + y^2 + 2gx + 2fy + c = 0$ as they are ON the circle:

(7,-2) => $(7)^2 + (-2)^2 + 2g(7) + 2f(-2) + c = 0$
=> $14g - 4f + c = -53$.......Eqn A
(4,5) => $(4)^2 + (-5)^2 + 2g(4) + 2f(5) + c = 0$
=> $8g + 10f + c = -41$.......Eqn B
(1,2) => $(1)^2 + (2)^2 + 2g(1) + 2f(2) + c = 0$
=> $2g + 4f + c = -5$...........Eqn C

- We solve the three equations A, B and C simultaneously to get:

$$f = -\frac{6}{5}, g = -\frac{24}{5}\ and\ c = \frac{47}{5}$$

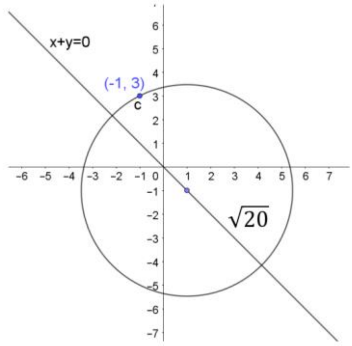
- Substitute these values back into the general equation above:

$$x^2 + y^2 + 2(-\frac{24}{5})x + 2(-\frac{6}{5})y + (\frac{47}{5}) = 0$$

- Finally, multiply through by 5 to kill the factions:
=> $5x^2 + 5y^2 - 48x - 12y + 47 = 0$

Example 2: A circle of radius length $\sqrt{20}$ contains the point (-1, 3). Its centre lies on the line x + y = 0. Find the equation of the two circles that satisfy these conditions.

- A diagram of this problem is:



- Firstly, we're told the radius is $\sqrt{20}$, so:
$\sqrt{g^2 + f^2 - c} = \sqrt{20}$
=> $g^2 + f^2 - c = 20$.........Eqn A (squaring both sides)
- Secondly, we're told (-1, 3) is on the circle, so:
(-1, 3) => $(-1)^2 + (3)^2 + 2g(-1) + 2f(3) + c = 0$
=> $-2g + 6f + c = -10$.......Eqn B
- And finally, we're told its centre is along the line x + y = 0, so:
$(-g) + (-f) = 0$
=> $g = -f$..........Eqn C
- To start solving, we can substitute equation C into equations A and B giving:
A: $(-f)^2 + f^2 - c = 20$ B: $-2(-f) + 6f + c = -10$
=> $2f^2 - c = 20$ => $8f + c = -10$
- Use eqn A to get 'c' on its own and substitute into eqn B:
A: $c = 2f^2 - 20$
=> B: $8f + 2f^2 - 20 = -10$
=> B: $f^2 + 4f - 5 = 0$ (dividing across by 2 and tidying up)
=> $(f + 5)(f - 1) = 0$
=> $f = -5$ or $f = 1$
- Now substitute our values for f into eqn B to find 'c':

If $f = -5$ => $8(-5) + c = -10$ => $c = 30$	If $f = 1$ => $8(1) + c = -10$ => $c = -18$

- And using eqn C above, we can find the values of 'g':

If $f = -5$ => $g = -(-5) = 5$	If $f = 1$ => $g = -1$

- So, the equations of the two circles will be:
$x^2 + y^2 + 10x - 10y + 30 = 0$ and $x^2 + y^2 - 2x + 2y - 18 = 0$