Topic 8: Difference Equations

1) Arithmetic Sequences/Series:

a) Linear(Arithmetic) Sequences:

Notes:

- A list of numbers where the **difference** between **each term** is the **same** every time. E.g. 3, 8, 13, 18, ………
- The General Term for an Arithmetic sequence is:

$$T_n = a + (n-1)d$$

See Tables pg 22

where 'a' is the first term and 'd' is the common difference between the terms.

b) Arithmetic Series:

Notes:

- If we add the terms of an arithmetic sequence together, then we get an arithmetic **series**.
- We need to be able to find the sum of the first n terms of such a series, which we can find using:

$$S_n = \frac{n}{2}\{2a + (n-1)d\}$$

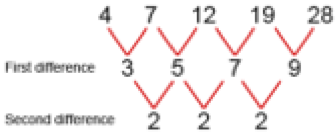
See Tables pg 22

where 'a' is the first term and 'd' is the common difference between the terms of the series.

2) Quadratic Sequences:

Notes:

- A sequence where the **second difference** is the **same** every time. E.g. 4, 7, 12, 19, 28……. (see below)



Steps to find General Term:

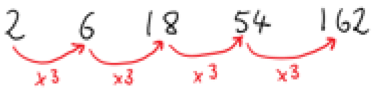
1. Let General Term = $T_n = an^2 + bn + c$
2. Find 2nd difference and let = 2a….solve for a.
3. Use any 2 terms to form two equations in b and c.
4. Solve both equations to find b and c.

3) Geometric Sequences/Series:

a) Geometric Sequences:

Notes:

- A sequence where each term is found by **multiplying** the previous term by the same number every time.



- The General Term for a Geometric sequence is:

$$T_n = a.r^{n-1}$$

See Tables pg 22

where 'a' is the first term and 'd' is the common difference.

b) Geometric Series:

Notes:

- If we add the terms of an geometric sequence together, then we get a geometric **series**.
- We need to be able to find the sum of the first n terms of such a series, which we can find using:

$$S_n = \frac{a(1-r^n)}{1-r}$$

See Tables pg 22

where 'a' is the first term and 'r' is the common ratio

4) Infinite Series/Limits of a Sequence:

a) Infinite Series:

Notes:

- Series where the terms of an infinite Geometric sequence are added up.

$$S_\infty = \frac{a}{1-r}$$

See Tables pg 22

(for an infinite Geometric Series, where $|r| < 1$)

c) Recurrence Relations:

Notes:

- A sequence which is defined showing how any term is connected to the previous term.

b) Limit of a Sequence:

- Sometimes a sequence can be approaching a particular number e.g. 1, $\frac{1}{2}$, $\frac{1}{4}$…… is a sequence that approaches 0.
- If a sequence approaches a certain number L, as the number of terms increases, then we say:

$$\lim_{n\to\infty} T_n = L$$

- Another very useful property of limits is:

$$\lim_{n\to\infty} \frac{1}{n^p} = 0$$

5) Solving Difference Equations:

a) 1st Order Difference Equations:

Notes:

- Equations with one term and a previous term e.g. $T_{n+1} = T_n + 5$
- For interest repayments, debt owing at the end of the term of the loan = 0

Steps for solving:

1. Let n = 1, 2, 3 and write out the first few terms: T_1, T_2, T_3
2. Leave in expanded form, to make it easier to spot the pattern
3. Watch for Geometric Series, that we can find the sum of, using the formula in 3(b) above

b) 2nd Order Difference Equations:

Notes:

- Equations with one term and the previous two terms e.g. $2u_{n+2} - 11u_{n+1} + 5u_n = 0$
- Two types:
  - Homogeneous: $2u_{n+2} - 11u_{n+1} + 5u_n = 0$
  - Inhomogeneous: $2u_{n+2} - 11u_{n+1} + 5u_n = 7n - 14$

c) 2nd Order Homogeneous Difference Equations:

Steps for solving:

1. Form characteristic equation i.e. $2x^2 - 11x + 5 = 0$ and solve.
2. Use the theorems below:

$$If\ roots\ are\ \boldsymbol{different}: \quad u_n = la^n + mb^n$$

$$If\ roots\ are\ \boldsymbol{same}: \quad u_n = la^n + mnb^n$$

3. Use terms given to evaluate l and m.
4. Write down solution.

d) 2nd Order Inhomogeneous Difference Equations:

Steps for solving:

1. Move all the terms that are related to each other to the LHS and leave other terms on the RHS. E.g. $2u_{n+2} - 11u_{n+1} + 5u_n = 7n$
2. Solve characteristic equation as in 5(b) to find **complimentary solution**.

E.g. for $2u_{n+2} - 11u_{n+1} + 5u_n = 7n - 14$ the characteristic equation is $2x^2 - 11x + 5 = 0$

2a. To find **particular solution**:

$$If\ RHS = Constant \Rightarrow u_n = a$$
$$If\ RHS = n \Rightarrow u_n = an + b$$
$$If\ RHS = n^2 \Rightarrow u_n = an^2 + bn + c$$
$$If\ RHS = a^n \Rightarrow u_n = k.a^n + b$$

2b. Get expressions for u_{n+1} and u_{n+2} and sub in to original equation to find the values of a and b.

4. Combine particular solution and complimentary solution to get the **total solution**.
5. Use terms given to evaluate l and m.
6. Write down solution.

6) General Tips for the Exam:

- Take care not to solve the simultaneous equations to find l and m until you have combined the particular and complimentary solutions.
- Make sure you know the two theorem results from 5(c).