End of Chapter Revision Sheet: Differential Equations 6th Year

Q1. Solve $\frac{dy}{dx} = 6y\sin x$, given $y = 1$ when $x = \pi$.

Q2. Solve $\frac{dy}{dx} = \sqrt{1 - y^2}$ if $y = 0$ when $x = 1$.

Q3. Solve the following differential equation: $x\frac{dy}{dx} = y - xy$, if $y = 3$ when $x = 5$.

Q4. Solve the following differential equation: $3\frac{dy}{dx} + \frac{x}{x^2+1} = 0$, if $y = 2$ when $x = 0$.

Q5. Solve $x\frac{dy}{dx} = (1 + 2x^2)y^2$, given that $y = 1$ when $x = 1$.

Q6. Solve the following differential equation: $\frac{dy}{dx} + y^2\cos x = 0$, if $y = 2$ when $x = \frac{\pi}{6}$.

Q7. $200\frac{dv}{dt} = 1600 + v^2$, if $v = 0$ when $t = 50$. Find v when $t = 55$.

Q8. $\frac{dv}{dt} = g - kv$ where g and k are constants. Given that $v = 0$ when $t = 0$, find the general solution.

Word Problems:

Q9. A body of mass $2\ kg$ is subjected to a retarding force of $6\sqrt{v}\ N$. At an instant its velocity is $9\ m/s$. Find how long is required for the body to come to rest.

Q10. At an instant a body has a velocity of $10\ m/s$ and is undergoing a retardation of $(1 + v)\ m/s^2$. How long does it take to come to rest?

Q11. The deceleration of a body is proportional to its speed. Find the constant of proportionality, k, if the speed decreases from $20\ m/s$ to $10\ m/s$ in $5\ s$.

Q12. A particle of mass m is thrown vertically upwards at $50\ m/s$ against a resistance of $0.05v^2\ N$ per unit mass. Find

(i) the maximum height reached by the particle

(ii) the time to reach the maximum height

(iii) the height and time as above, if air resistance is ignored

Q13. A car of mass $600\ kg$ has an engine that works at a constant rate of $42\ kW$. Find the time taken to accelerate from $10\ m/s$ to $30\ m/s$.

Q14. The population in a region is decreasing at t rate of 2% every 4 years. If the population of the region was $150{,}000$ in 2000, what would you expect its population to be in 2025, based on this rate of decrease?

Q15. The initial mass of a radioactive isotope was $600\ g$. The half-life of the isotope is 10 days (this is the time taken for half of the mass to decay radioactively). Find the mass of the isotope after 50 days.

Q16. A car of mass $600\ kg$ moves along a flat horizontal road against a resistance of $150v\ N$, where v is the speed of the car in m/s. The engine has a constant power output of $120kW$.

(i) Show that the equation of motion of the car is $4\frac{dv}{dt} = \frac{800 - v^2}{v}$.

(ii) Calculate the time it takes the car to accelerate from $20\ m/s$ to $25\ m/s$.

(iii) Find the maximum speed that the car can reach.

Answers:

<table>
<tr><td>Q1. $y = e^{-6(\cos x + 1)}$</td><td>Q2. $y = \sin(x - 1)$</td><td>Q3. $y = \frac{3}{5}x.e^{5-x}$</td><td>Q4. $y = -\frac{1}{6}\ln(x^2+1) + 2$</td></tr>
<tr><td>Q5. $y = \frac{1}{2 - \ln x - x^2}$</td><td>Q6. $y = \frac{1}{\sin x}$</td><td>Q7. 62.296</td><td>Q8. $v = \frac{g - ge^{-kt}}{k}$</td></tr>
<tr><td>Q9. $t = 2\ s$</td><td>Q10. $t = 2.4\ s$</td><td>Q11. $k = 0.1386$ or $\frac{\ln 2}{5}$</td><td></td></tr>
<tr><td colspan="2">Q12. (i) $26.21\ m$ (ii) $1.86\ s$ (iii) $127.55\ m, 5.1\ s$</td><td>Q13. $\frac{400}{7}\ s$ or $57.14\ s$</td><td>Q14. $132{,}209$</td></tr>
<tr><td colspan="2">Q15. $18.76\ g$</td><td colspan="2">Q16. (ii) $1.65\ s$ (iii) $20\sqrt{2}\ m/s$ or $28.28\ m/s$</td></tr>
</table>

Past Exam Questions:

SEC HL Sample Q3(a)

A particle has initial displacement s_0 from a fixed point P. It moves away from P with initial velocity u and constant acceleration $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$.

Use calculus to derive an expression for s, the displacement of the particle from P at any time t.

2024 Q6(b)

(b) The acceleration of a particle moving in a straight line may be expressed in terms of its velocity v in m s^{-1} and its displacement s in m by the differential equation:

$$v\frac{dv}{ds} = e^{\frac{v^2}{4}}$$

(i) Solve the differential equation to find an expression for v in terms of s, given that $v = 0$ when $s = 0$.

(ii) Calculate the velocity of the particle when $s = 0.3$ m.

2023 Deferred Q4

In 1838 the Belgian mathematician Pierre François Verhulst published a differential equation to model rate of change of population P with respect to time t:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

where r and K are constants for a given population.

For a certain species of insect in an environment it is known that the population can increase by up to 8% per week, i.e. $r = 0.08$.

At $t = 0$ weeks there are 20 insects in the population.

When the population P is small relative to K, the ratio $\frac{P}{K}$ is also small and Verhulst's model can be approximated by the simplified differential equation:

$$\frac{dP}{dt} = rP$$

(i) Solve this simplified differential equation to find an expression for P in terms of t.

(ii) Calculate P to the nearest whole number when $t = 12$ weeks.

(iii) Explain why this approximation of Verhulst's model is not practical for predicting the long-term behaviour of the population of insects.

(iv) Solve the differential equation for Verhulst's model:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

to find an expression that relates P, K and t.

Note that $\frac{1}{y(x-y)} = \frac{1}{x}\left(\frac{1}{y} + \frac{1}{x-y}\right)$.

Past Exam Questions:

Sample Paper: $s = ut + \frac{1}{2}at^2 + s_0$	2024: (i) $v = \sqrt{4\ln(\frac{2}{2-s})}$ (ii) $0.806\ m/s$
2023 Deferred: (i) $P = 20e^{0.08t}$ (ii) 52 $weeks$ (iii) See worked solutions (iv) $P = \frac{20Ke^{0.08t}}{K - 20 + 20e^{0.08t}}$	