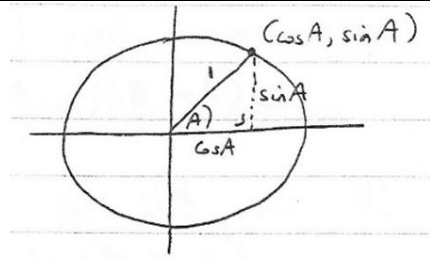
Proof of Trig Identities:

1) To prove: $\cos^2 A + \sin^2 A = 1$

Using Pythagoras Theorem:

$$(\cos A)^2 + (\sin A)^2 = (1)^2$$

=> $\cos^2 A + \sin^2 A = 1$



2) To prove: $\cos(A - B) = \cos A \cos B + \sin A \sin B$

Let $P(\cos A, \sin A)$ and $Q(\cos B, \sin B)$ be two points on a unit circle.

Using distance formula:

$$|PQ| = \sqrt{(\cos A - \cos B)^2 + (\sin A - \sin B)^2}$$

$$|PQ|^2 = \cos^2 A + \cos^2 B - 2\cos A \cos B + \sin^2 A + \sin^2 B - 2\sin A \sin B$$

$$= (\cos^2 A + \sin^2 A) + (\cos^2 B + \sin^2 B) - 2(\cos A \cos B + \sin A \sin B)$$

$$= 1 + 1 - 2(\cos A \cos B + \sin A \sin B)$$

$$= 2 - 2(\cos A \cos B + \sin A \sin B)$$

Using Cosine Rule to find |PQ| instead:

$$|PQ|^2 = (1)^2 + (1)^2 - 2(1)(1)\cos(A - B)$$

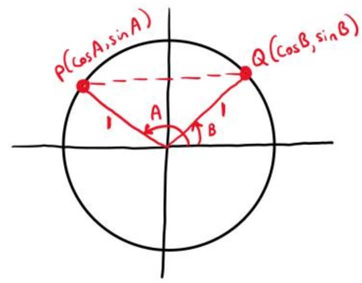
=> $|PQ|^2 = 2 - 2\cos(A - B)$

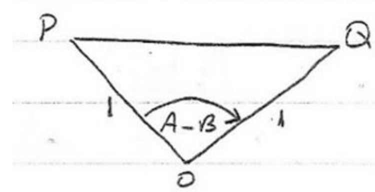
Equating the two expressions for $|PQ|^2$:

$$2 - 2\cos(A - B) = 2 - 2(\cos A \cos B + \sin A \sin B) \qquad (-2)$$

=> $-2\cos(A - B) = -2(\cos A \cos B + \sin A \sin B) \qquad (\div -2)$

=> $\cos(A - B) = \cos A \cos B + \sin A \sin B$





3) To prove: $\cos(A + B) = \cos A \cos B - \sin A \sin B$

We know from (2) that:

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

If we fill in $-B$ instead of B:

$$\cos\big(A - (-B)\big) = \cos A \cos(-B) + \sin A \sin(-B)$$

=> $\cos(A + B) = \cos A \cos(-B) - \sin A \sin(-B)$

Since $\cos(-B) = \cos(B)$ and $\sin(-B) = -\sin(B)$

=> $\cos(A + B) = \cos A \cos B + \sin A \sin B$

4) To prove: $\sin(A+B) = \sin A \cos B + \cos A \sin B$

We know from (2) that $\cos(A-B) = \cos A \cos B + \sin A \sin B$

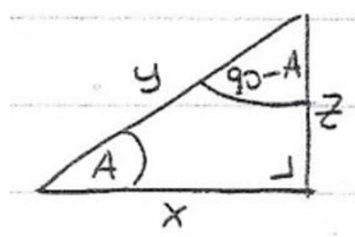
If we fill in $90 - A$ instead of A, we get:

$$\cos((90-A)-B) = \cos(90-A)\cos B + \sin(90-A)\sin B$$

From the diagram on the right:

$$\sin(90-A) = \frac{x}{y} = \cos A$$

$$\cos(90-A) = \frac{z}{y} = \sin A$$



=> $\cos((90-A)-B) = \sin A \cos B + \cos A \sin B$

=> $\cos\big(90-(A+B)\big) = \sin A \cos B + \cos A \sin B$

=> $\sin(A+B) = \sin A \cos B + \cos A \sin B$

5) To prove: $\sin(A-B) = \sin A \cos B - \cos A \sin B$

We know from (4) that $\sin(A+B) = \sin A \cos B + \cos A \sin B$

If we fill in $-B$ instead of B:

$$\sin(A+(-B)) = \sin A \cos(-B) + \cos A \sin(-B)$$

=> $\sin(A-B) = \sin A \cos(-B) + \cos A \sin(-B)$

Since $\cos(-B) = \cos(B)$ and $\sin(-B) = -\sin(B)$

=> $\sin(A-B) = \sin A \cos B - \cos A \sin B$

6) To prove: $\cos 2A = \cos^2 A - \sin^2 A$

We know from (3) that $\cos(A+B) = \cos A \cos B - \sin A \sin B$

If we replace B by A we get:

$$\cos(A+A) = \cos A \cos A - \sin A \sin A$$

=> $\cos 2A = \cos^2 A - \sin^2 A$

7) To prove: $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B + \sin A \sin B}$$ Using (3) and (4)

We now divide each of the four terms by $\cos A \cos B$:

$$\tan(A+B) = \frac{\frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} + \frac{\sin A \sin B}{\cos A \cos B}}$$

=> $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

8) To prove: Sine Rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Area of the triangle shown = $\frac{1}{2}ab\sin C$

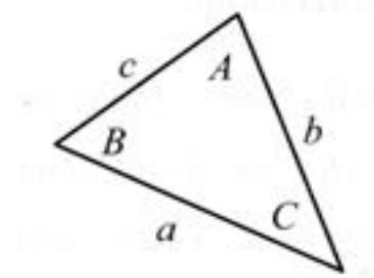
$$\frac{1}{2}ab\sin C = \frac{1}{2}ac\sin B = \frac{1}{2}bc\sin A$$

Dividing all three by $\frac{1}{2}abc$ gives:

$$\frac{\frac{1}{2}ab\sin C}{\frac{1}{2}abc} = \frac{\frac{1}{2}ac\sin B}{\frac{1}{2}abc} = \frac{\frac{1}{2}bc\sin A}{\frac{1}{2}abc}$$

=> $\frac{\sin C}{c} = \frac{\sin B}{b} = \frac{\sin A}{a}$

=> $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$



9) To prove: Cosine Rule: $a^2 = b^2 + c^2 - 2bc\cos A$

Consider the triangle shown on the right.

Using the distance formula to find a:

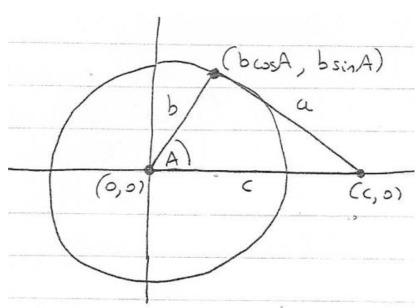
$$a = \sqrt{(b\cos A - c)^2 + (b\sin A - 0)^2}$$

=> $a^2 = b^2\cos^2 A + c^2 - 2bc\cos A + b^2\sin^2 A$

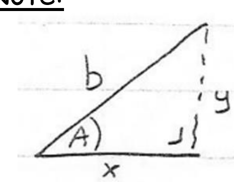
=> $a^2 = b^2(\cos^2 A + \sin^2 A) + c^2 - 2bc\cos A$

=> $a^2 = b^2(1) + c^2 - 2bc\cos A$

=> $a^2 = b^2 + c^2 - 2bc\cos A$



Note:



$$\cos A = \frac{x}{b}$$

=> $x = b\cos A$

$$\sin A = \frac{y}{b}$$

=> $y = b\sin A$