Topic 2: Complex Numbers

1) The Basics:

a) Definition:

Notes:

- A complex number is of the form $a + bi$.
- $i = \sqrt{-1}$

$$i^2 = -1$$

b) Adding/Subtracting:

Note:

- Add/Subtract like terms together as we did in Algebra

Example: $z_1 = 3 - 2i$ and $z_2 = 4 + 5i$, evaluate i) $z_1 + z_2$ ii) $z_1 - 2z_2$

i) $z_1 + z_2 = 3 - 2i + 4 + 5i = 7 + 3i$

i) $z_1 - 2z_2 = 3 - 2i - 2(4 + 5i) = 3 - 2i - 8 - 10i = -5 - 12i$

c) Multiplying Complex Numbers:

Notes:

- When multiplying by a constant, we treat as usual
- When multiplying by an 'i' or another complex number, we have to use the definition o.

Example 1: If $z_1 = 2 - 5i$, evaluate $3z_1$.

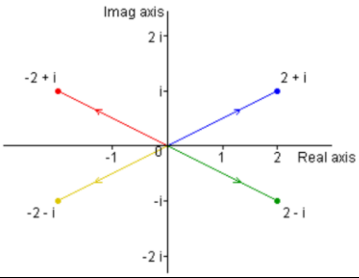
$3z_1 = 3(2 - 5i)$
$= 6 - 15i$

Example 2: If $z_1 = 3 + 4i$ and $z_2 = -2 - 5i$, evaluate $z_1.z_2$.

$z_1.z_2 = (3 + 4i)(-2 - 5i)$
$= 3(-2 - 5i) + 4i(-2 - 5i)$
$= -6 - 15i - 8i - 20i^2$
$= -6 - 15i - 8i - 20(-1)$
$= -6 - 15i - 8i + 20$
$= 14 - 23i$

2) Argand Diagram/Modulus/Conjugate:

a) Argand Diagram:



b) Conjugate:

If $z = a + bi$
=> Conjugate = $\bar{z} = a - bi$

Examples:

i) If $z_1 = 3 + 4i$ => $\bar{z_1} = 3 - 4i$

ii) If $z_2 = -2 - 5i$ => $\bar{z_2} = -2 + 5i$

iii) If $z_3 = 3i - 1$ => $\bar{z_3} = -3i - 1$

c) Modulus:

Notes:

- Modulus is the distance from the origin (0, 0) to the complex number

If $z = a + bi$
$|z| = \sqrt{a^2 + b^2}$

Example: If $z_1 = 3 + 4i$ and $z_2 = -2 - 5i$, evaluate i) $|z_2|$ and ii) $|3z_2 - 2z_1|$

i) $|z_1| = \sqrt{(-2)^2 + (-5)^2}$
$= \sqrt{29}$

ii) Before getting the modulus, we need to tidy up the expression into the form a + bi:

$3z_2 - 2z_1 = 3(-2 - 5i) - 2(3 + 4i)$
$= -6 - 15i - 6 - 8i = -12 - 23i$

=> $|3z_2 - 2z_1| = |-12 - 23i| = \sqrt{(-12)^2 + (-23)^2} = \sqrt{673} = 25.9$

3) Division of Complex Numbers/Quadratic Equations with Complex Roots:

a) Division of Complex Numbers:

Multiply above and below by the conjugate of the denominator.

Example: Evaluate $\frac{3 + 5i}{2 - 3i}$

$$\frac{3 + 5i}{2 - 3i} \times \frac{2 + 3i}{2 + 3i}$$

$$= \frac{(3 + 5i)(2 + 3i)}{(2 - 3i)(2 + 3i)}$$

$$= \frac{3(2 + 3i) + 5i(2 + 3i)}{2(2 + 3i) - 3i(2 + 3i)} = \frac{6 + 9i + 10i + 15i^2}{4 + 6i - 6i - 9i^2}$$

$$= \frac{6 + 9i + 10i - 15}{4 + 6i - 6i + 9}$$

$$= \frac{-9 + 19i}{13} = -\frac{9}{13} + \frac{19}{13}i$$

b) Quadratic Equations with Complex Roots:

- To solve an equation of the form $az^2 + bz + c = 0$, we can use the '-b Formula' from Algebra Topic – Section 3d

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example: Solve the equation $z^2 - 2z + 10 = 0$.

=> a = 1, b = -2 and c = +10

$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$z = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(10)}}{2(1)}$

$z = \frac{2 \pm \sqrt{-36}}{2}$ (tidying up under the $\sqrt{\ }$)

$z = \frac{2 \pm \sqrt{-1}\sqrt{36}}{2}$ (as $\sqrt{ab} = \sqrt{a}\sqrt{b}$)

$z = \frac{2 \pm 6i}{2}$ (using definition of i from Section 1a)

$z = 1 \pm 3i$