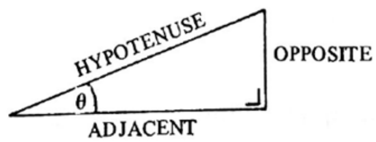
- Chapter 3: Projectiles
- Topic 18: Intro and Equations of Motion

- Resolving Vectors:
  - We learned in chapter 1 how to resolve a vector into horizontal and vertical components.
  - Remember:

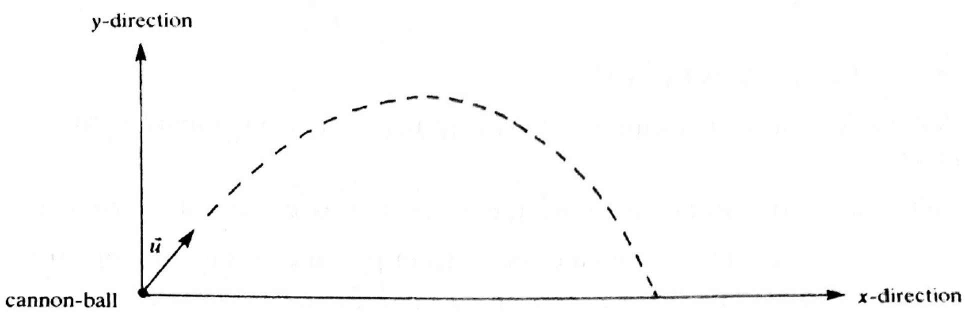
In a right-angled triangle:



Adjacent = $h\cos\theta$

Opposite = $h\sin\theta$

  - So, we can replace any vector with two component vectors, one in the horizontal direction, and the other, in the vertical direction.
- Intro to Projectiles:
  - We saw in Topic 13 that objects moving freely under gravity accelerate uniformly at 9.8m/s^2.
  - Any object that moves freely under gravity is known as a projectile.
  - As all projectiles are moving freely under gravity, we can use the equations of motion developed in chapter 2, to analyse projectiles.
  - Examples of projectiles would include a cannon ball or a bullet from a gun.
  - When examining the motion of projectiles, we look at the distance travelled, and the velocity in both the horizontal (X) direction and the vertical (Y) direction.



- Forming Equations of Motion:
  - The first equation we derived was:

$$v = u + at$$ Equation 1

  - If we now look at the horizontal (X) direction only, we can easily adjust our equation as follows:

$$v_x = u_x + a_x t$$

- As gravity only acts in the vertical direction, there will never be any acceleration in the X direction, so our equation simplifies to:

$$v_x = u_x$$

- If we now look at velocity in the y direction:

$$v_y = u_y + a_y t$$

- In this case, the only acceleration in the Y direction will be gravity itself, so we can substitute 9.8 or 'g' into our equation to represent that.
- As the gravity always acts downwards, we have to use a '-' sign when substituting in 'g' however:

$$v_y = u_y - gt$$

- In a similar way, we can derive two equations for distance travelled in the X and Y direction, and they are:

$$s_x = u_x t$$
$$s_y = u_y t - \frac{1}{2} g t^2$$

- In summary, we have now derived 4 very important equations that we can use for analysing projectile motion:

$$v_x = u_x$$
$$v_y = u_y - gt$$
$$s_x = u_x t$$
$$s_y = u_y t - \frac{1}{2} g t^2$$

- Example: Pg 53 Ex 3C Q1

A particle is projected with an initial velocity of $30\vec{i} + 70\vec{j}$ from a point on a horizontal plane. Find:

i) The time it takes to reach the maximum height
ii) The maximum height reached
iii) The range
iv) The two times when the particle is travelling 50 m/s

Solution:

Method 1: Table and Graph

i)

- Assuming the acceleration due to gravity is $10 m/s^2$, we will look at a table of the velocity of the projectile over the first few seconds of its motion.
- As we saw above, gravity only acts in the vertical direction at a rate of $10\ m/s^2$, so the projectile will lose 10m/s in the vertical direction every second:

Time	Velocity		Time	Velocity
t = 0	$30\vec{i} + 70\vec{j}$		t = 8	$30\vec{i} - 10\vec{j}$
t = 1	$30\vec{i} + 60\vec{j}$		t = 9	$30\vec{i} - 20\vec{j}$
t = 2	$30\vec{i} + 50\vec{j}$		t = 10	$30\vec{i} - 30\vec{j}$
t = 3	$30\vec{i} + 40\vec{j}$		t = 11	$30\vec{i} - 40\vec{j}$
t = 4	$30\vec{i} + 30\vec{j}$		t = 12	$30\vec{i} - 50\vec{j}$
t = 5	$30\vec{i} + 20\vec{j}$		t = 13	$30\vec{i} - 60\vec{j}$
t = 6	$30\vec{i} + 10\vec{j}$		t = 14	$30\vec{i} - 70\vec{j}$
t = 7	$30\vec{i} + 0\vec{j}$			

Things to draw their attention to:
i) the symmetry of motion
ii) how do we spot time to max height?
iii) speed of landing = speed of take-off –> show calculation of speeds to prove to them

- So, from our table above, we can see the time taken to reach max height is 7 secs. (as the velocity in the J direction at that time is 0)

ii)

- We can now calculate the position of the projectile at regular time intervals along its path:

In the 1st second: $s = \left(\frac{u+v}{2}\right)t$

$$= \left(\frac{30\vec{i} + 70\vec{j} + 30\vec{i} + 60\vec{j}}{2}\right)(1) = 30\vec{i} + 65\vec{j}$$

=> The particle will move 30m to the right and up 65m

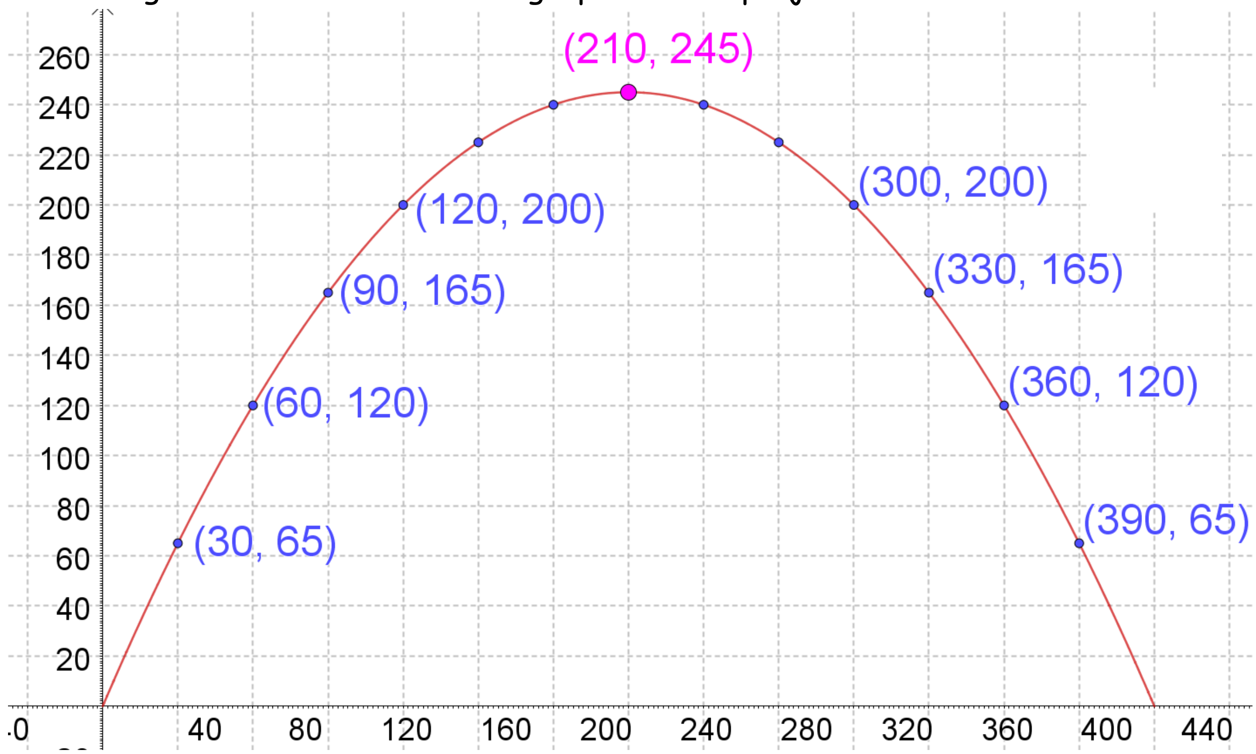
In the 2nd second: $s = \left(\frac{u+v}{2}\right)t$

$$= \left(\frac{30\vec{i} + 60\vec{j} + 30\vec{i} + 50\vec{j}}{2}\right)(1) = 30\vec{i} + 55\vec{j}$$

=> The particle will move another 30m to the right and up 55m from where it was

Put up the Geogebra graph after the 1st calculation and ask them to do the second calculation and to puzzle out where the projectile has moved to? Show them then on Geogebra. Maybe get them to do the 3rd second then to consolidate the idea.

- Continuing in the same manner the graph for the projectile's motion will be:



- From our graph we can see that the max height is: 245m

iii)

- The range is the horizontal distance travelled before the projectile hits the ground.
- We know from the table above that the projectile was in motion for 14 seconds and it's travelling 30m/s to the right every second, so the range will be 14 x 8 = 112m

iv)

- If the speed of the particle is 50 m/s, then $|a\vec{i} + b\vec{j}| = 50$

  => $\sqrt{a^2 + b^2} = 50$

  => $a^2 + b^2 = 2500$ (squaring both sides)

- So, we need to look for a and b values that when squared and added together, will give 2500 i.e. $30\vec{i} + 40\vec{j}$ and $30\vec{i} - 40\vec{j}$
- These velocities occur when t = 3 secs and when t = 11 secs

Method 2: Equations of Motion

- We will begin by forming our 4 equations of motion.
- As the initial velocity is $30\vec{i} + 70\vec{j}$, that means that it starts out by travelling 30m/s in the X direction and 70m/s in the Y direction, so these values represent u_x and u_y respectively.
- So:

$v_x = u_x = 30$	$v_y = u_y - gt = 70 - 10t$
$s_x = u_x t = 30t$	$s_y = u_y t - \frac{1}{2}gt^2 = 70t - \frac{1}{2}(10)t^2$ $= 70t - 5t^2$

i) At greatest height, the projectile it is stopped for a fraction of a second and is not ascending or descending. That means that the velocity in the Y direction must be 0:

$v_y = 70 - 10t$

=> $70 - 10t = 0$

=> $70 = 10t$

=> $t = \frac{70}{10} = 7\ seconds$

ii) We now know that the particle reached its greatest height after 7 seconds, so we can use our expression for s_y now, to find what the greatest height actually is:

$$s_y = 70t - 5t^2$$
$$= 70(7) - (5)(7)^2$$
$$= 490 - 245$$
$$= 245m$$

iii) When the particle hits the ground, we know that the distance travelled in the Y direction will be 0 i.e. $s_y = 0$

$s_y = 70t - 5t^2$

=> $70t - 5t^2 = 0$

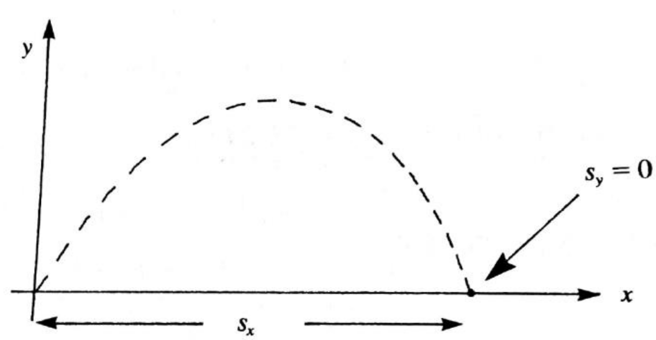
=> $5t^2 - 70t = 0$ …………………(rearranging to put t^2 term first)

=> $5t(t - 14) = 0$

=> $5t = 0\ \ OR\ \ t - 14 = 0$

=> $t = 0\ \ OR\ \ t = 14$

- We have got two answers for t, which makes sense if we look at the diagram of the particle's motion again:



- We solved where $s_y = 0$, which is when the particle hits the ground, but it would also be at the very start, before we launch the projectile.
- To finish this part, we now need to use our value for t, to find the horizontal distance travelled in that time i.e. s_x:

$$s_x = 10t$$
$$= 30(\ 14\ )$$
$$= 420\ m$$

iv) For this part, we know the velocity at ANY point is given by:

$$v_x\vec{\imath} + v_y\vec{\jmath} = 30\vec{\imath} + (70 - 10t)\vec{\jmath}$$

- If the speed of the projectile is 50 m/s, then $|v_x\vec{\imath} + v_y\vec{\jmath}|$ = 50, so:

$$\sqrt{(30)^2 + (70 - 10t)^2} = 50$$

=> $\sqrt{900 + 4900 - 1400t + 100t^2} = 50$ (expanding the brackets)

=> $100t^2 - 1400t + 5800 = 2500$ (squaring both sides)

=> $100t^2 - 1400t + 3300 = 0$

=> $t^2 - 14t + 33 = 0$ (dividing through by 100)

=> $(t - 3)(t - 11) = 0$

=> $t - 3 = 0$ or $t - 11 = 0$

=> $t = 3$ secs or $t = 11$ secs

Day 1 : Classwork Questions: pg 53/54 Ex 3C Qs 2/3/6/7

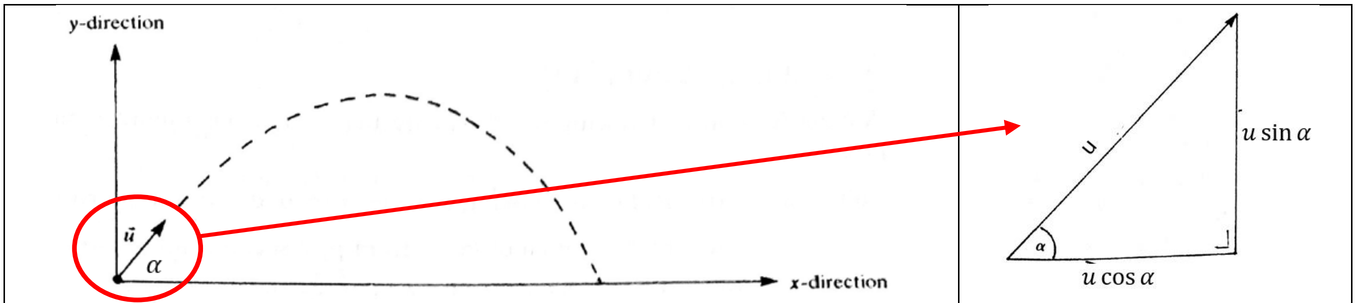
Day 2 : Classwork Questions: pg 56/57 Ex 3D Qs 2/4/5/10 and then try Q9/12

- Topic 19: Maximum Range

- Example: A particle is fired with initial speed u. If it is fired at an angle α to the horizontal plane from a point on a horizontal plane, find an expression for (i) the greatest height (ii) the range (iii) the maximum range and (iv) the angle of projection that will give max range.

Solution:

- We will begin by writing a general expression for the initial velocity:



Using the diagram above, the initial velocity is:

$$= u_x\vec{i} + u_y\vec{j}$$
$$= u\cos\alpha\,\vec{i} + u\sin\alpha\,\vec{j}$$

- We can now form our 4 equations of motion:

$$v_x = u_x = u\cos\alpha$$
$$v_y = u_y - gt$$
$$= u\sin\alpha - gt$$
$$s_x = u_x t$$
$$= u\cos\alpha\, t$$
$$s_y = u_y t - \frac{1}{2}gt^2$$
$$= u\sin\alpha\, t - \frac{1}{2}gt^2$$

i) As we saw in Example 1, at the greatest height, the velocity in the Y direction will be 0 i.e. $v_y = 0$: $v_y = u\sin\alpha - 9.8t = 0$

=> $9.8t = u\sin\alpha$

=> $t = \frac{u\sin\alpha}{g}$

- We can now use this time, to find the greatest height s_y above the plane:

$$s_y = u\sin\alpha\, t - \frac{1}{2}gt^2$$

$= u\sin\alpha\,(\frac{u\sin\alpha}{g}) - \frac{1}{2}g(\frac{u\sin\alpha}{g})^2$.................(filling in for t)

$= \frac{u^2\sin^2\alpha}{g} - \frac{1}{2}g\,(\frac{u^2\sin^2\alpha}{g^2})$......(squaring the time in the 2nd term)

$= \frac{u^2\sin^2\alpha}{g} - \frac{u^2\sin^2\alpha}{2g}$...............(cancelling the g into the g^2)

$= \frac{2u^2\sin^2\alpha}{2g} - \frac{u^2\sin^2\alpha}{2g}$............(getting 1st term into the same denominator)

$= \frac{2u^2\sin^2\alpha - u^2\sin^2\alpha}{2g}$................(writing as a single fraction)

$= \frac{u^2\sin^2\alpha}{2g}$................................(tidying up the numerator)

ii) Again, as we saw in Example 1 before, the range is given by s_x when $s_y = 0$.

$$s_y = u\sin\alpha\, t - \frac{1}{2}gt^2 = 0$$

=> $t(u\sin\alpha - \frac{1}{2}gt) = 0$

=> $t = 0 \quad OR \quad u\sin\alpha - \frac{1}{2}gt = 0$

=> $t = 0 \quad OR \quad \frac{1}{2}gt = u\sin\alpha$

=> $t = 0 \quad OR \quad gt = 2u\sin\alpha$

=> $t = 0 \quad OR \quad t = \frac{2u\sin\alpha}{g}$

- We now find an expression for s_x, using our expression for t:

$s_x = u_x t$

$= u\cos\alpha\, t$

$= u\cos\alpha\,(\frac{2u\sin\alpha}{g})$

$= \frac{2u^2\sin\alpha\cos\alpha}{g}$

$= \frac{u^2\sin 2\alpha}{g}$...(using an identity from pg14 of the Tables book i.e. sin2A = 2sinAcosA)

iii) The final part of this question uses a property of the Sine function, that you wouldn't have come across yet on your Maths course, but you will in due course.

- The biggest value a Sine function can take is 1, so the range will be at a maximum when the $\sin 2\alpha = 1$.
- We can see fill this in to our expression from the previous part, to get an expression for the maximum range:

$R_{max} = \frac{u^2\sin 2\alpha}{g} = \frac{u^2(1)}{g} = \frac{u^2}{g}$

iv) We said in the previous part, that the biggest value the Sine function can take is 1, so we can use this to solve for the angle α :

$\sin 2\alpha = 1$

=> $2\alpha = \sin^{-1} 1$

=> $2\alpha = 90°$

=> $\alpha = 45°$

Will need to show them how to finish the solving when they get as far as Height = Range

Day 1 : Classwork Questions: pg 58/59 Ex 3E Qs 3/7/13/14/15/20

- Topic 20: Target Practice:
  - When dealing with projectiles that hit a precise target, there are two important formulae from Trigonometry that we can use:

$$\tan A = \frac{\sin A}{\cos A} \quad \text{and} \quad 1 + \tan^2 A = \frac{1}{\cos^2 A}$$

- **Example:** Pg 60 Ex 3F Q1

  A particle is projected from a point O on horizontal ground at an angle A to the horizontal. The initial speed is $\sqrt{24g}$ m/s. It hits a small target whose position vector relative to O is $16\vec{\imath} + \frac{11}{3}\vec{\jmath}$ metres. Find two values of A to the nearest degree.

  Solution:
  - As always, we will form our equations of motion first:

$$v_x = u_x = \sqrt{24g}\cos A$$
$$v_y = u_y - gt$$
$$= \sqrt{24g}\sin A - gt$$
$$s_x = u_x t$$
$$= \sqrt{24g}\cos A\, t$$
$$s_y = u_y t - \frac{1}{2}gt^2$$
$$= \sqrt{24g}\sin A\, t - \frac{1}{2}gt^2$$

  - Right at the moment when the particle strikes the target $S_x = 16$ and $S_y = \frac{11}{3}$ so:

$$\sqrt{24g}\cos A\, t = 16 \quad \text{and} \quad \sqrt{24g}\sin A\, t - \frac{1}{2}gt^2 = \frac{11}{3}$$

  - We now use the left-hand equation to find an expression for t:

$$\sqrt{24g}\cos A\, t = 16$$
$$\Rightarrow t = \frac{16}{\sqrt{24g}\cos A}$$

  - and we can substitute that into the second equation then:

$$\sqrt{24g}\sin A\, t - \frac{1}{2}gt^2 = \frac{11}{3}$$
$$\Rightarrow \sqrt{24g}\sin A\left(\frac{16}{\sqrt{24g}\cos A}\right) - \frac{1}{2}g\left(\frac{16}{\sqrt{24g}\cos A}\right)^2 = \frac{11}{3} \text{.....(filling in for t)}$$
$$\Rightarrow \sqrt{24g}\sin A\left(\frac{16}{\sqrt{24g}\cos A}\right) - \frac{1}{2}g\left(\frac{256}{24g\cos^2 A}\right) = \frac{11}{3} \text{......(squaring t and cancelling 'g')}$$
$$\Rightarrow \frac{16\sin A}{\cos A} - \frac{16}{3\cos^2 A} = \frac{11}{3} \text{..........(multiplying fractions)}$$
$$\Rightarrow 16\left(\frac{\sin A}{\cos A}\right) - \frac{16}{3}\left(\frac{1}{\cos^2 A}\right) = \frac{11}{3} \text{..........(isolating the trig parts)}$$
$$\Rightarrow 16(\tan A) - \frac{16}{3}(1 + \tan^2 A) = \frac{11}{3} \text{..........(using the identities above)}$$
$$\Rightarrow 48(\tan A) - 16(1 + \tan^2 A) = 11 \text{..........(multiplying across by 3)}$$
$$\Rightarrow 48\tan A - 16 - 16\tan^2 A) = 11$$

=> $16\tan^2 A - 48\tan A + 16 = -11$…………………(rearranging to put $\tan^2 A$ first)

=> $16\tan^2 A - 48\tan A + 27 = 0$

=> $(4\tan A - 9)(4\tan A - 3) = 0$……………………(factorising the quadratic)

=> $4\tan A - 9 = 0 \quad OR \quad 4\tan A - 3 = 0$

=> $\tan A = \frac{9}{4} \quad OR \quad \tan A = \frac{3}{4}$

=> $A = \tan^{-1}\frac{9}{4} \quad OR \quad A = \tan^{-1}\frac{3}{4}$

=> $A = 66° \quad OR \quad A = 37°$

Day 1: Classwork Questions: pg 60/61 Ex 3E Qs 2/3/4 and then try Qs 6 if okay on those

Day 2: Classwork Questions: pg 60/61 Ex 3E Qs 9/11/14 and then try Q15/19

Revision Questions and Test