Worked Solutions

Q1.

i) $u_n = 3u_{n-1} + 5 \quad u_0 = 2$

$n=1$: $u_1 = 3u_0 + 5 = 3(2) + 5$

$n=2$: $u_2 = 3u_1 + 5 = 3(3(2)+5) + 5 = (3)^2(2) + 3(5) + 5$

$n=3$: $u_3 = 3u_2 + 5 = 3((3)^2(2) + 3(5) + 5) + 5 = (3)^3(2) + (3)^2(5) + 3(5) + 5$

$a = 5 \quad r = 3$

$$\Rightarrow u_n = (3)^n(2) + \frac{5(1-(3)^n)}{1-3}$$

$$= 2(3)^n - \frac{5}{2}(1-(3)^n)$$

$$= 2(3)^n - \frac{5}{2} + \frac{5}{2}(3)^n$$

$$= \boxed{\frac{9}{2}(3)^n - \frac{5}{2}}$$

ii) $u_n = \frac{1}{4}u_{n-1} + 2, \quad u_0 = 12$

$n=1$: $u_1 = \frac{1}{4}u_0 + 2 = \frac{1}{4}(12) + 2$

$n=2$: $u_2 = \frac{1}{4}(u_1) + 2 = \frac{1}{4}\left(\frac{1}{4}(12)+2\right) + 2 = \left(\frac{1}{4}\right)^2(12) + \frac{1}{4}(2) + 2$

$n=3$: $u_3 = \frac{1}{4}u_2 + 2 = \frac{1}{4}\left(\left(\frac{1}{4}\right)^2(12) + \frac{1}{4}(2) + 2\right) + 2 = \left(\frac{1}{4}\right)^3(12) + \left(\frac{1}{4}\right)^2(2) + \frac{1}{4}(2) + 2$

$a = 2 \quad r = \frac{1}{4}$

$$\Rightarrow u_n = \left(\frac{1}{4}\right)^n(12) + \frac{2\left(1-\left(\frac{1}{4}\right)^n\right)}{1-\frac{1}{4}}$$

$$= 12\left(\frac{1}{4}\right)^n + \frac{8}{3}\left(1-\left(\frac{1}{4}\right)^n\right)$$

$$= \left(12 - \frac{8}{3}\right)\left(\frac{1}{4}\right)^n + \frac{8}{3}$$

$$= \boxed{\frac{28}{3}\left(\frac{1}{4}\right)^n + \frac{8}{3}}$$

iii) $u_n = -2u_{n-1} + 7 \qquad u_1 = 8$

$n=1$

$u_1 = -2u_0 + 7$

$8 = -2u_0 + 7$

$2u_0 = -1$

$u_0 = -\frac{1}{2}$

$\Rightarrow u_1 = -2\left(-\frac{1}{2}\right) + 7$

$n=2$

$u_2 = -2u_1 + 7$

$= -2\left(-2\left(-\frac{1}{2}\right) + 7\right) + 7$

$= (-2)^2\left(-\frac{1}{2}\right) - 2(7) + 7$

$n=3$

$u_3 = -2u_2 + 7$

$= -2\left((-2)^2\left(-\frac{1}{2}\right) - 2(7) + 7\right) + 7$

$= (-2)^3\left(-\frac{1}{2}\right) + (2)^2(7) - 2(7) + 7$

$a = 7 \quad r = -2$

$$\Rightarrow u_n = (-2)^n\left(-\frac{1}{2}\right) + \frac{7\left(1 - (-2)^n\right)}{1 - (-2)}$$

$$= -\frac{1}{2}(-2)^n + \frac{7}{3}\left(1 - (-2)^n\right)$$

$$= \left(-\frac{1}{2} - \frac{7}{3}\right)(-2)^n + \frac{7}{3}$$

$$= \boxed{-\frac{17}{6}(-2)^n + \frac{7}{3}}$$

iv) $u_n = 7u_{n-1} - 30, \quad u_0 = 4$

$n=1$

$u_1 = 7u_0 - 30$

$= 7(4) - 30$

$n=2$

$u_2 = 7u_1 - 30$

$= 7\left(7(4) - 30\right) - 30$

$= (7)^2(4) - 7(30) - 30$

$n=3$

$u_3 = 7u_2 - 30$

$= 7\left((7)^2(4) - 7(30) - 30\right) - 30$

$= (7)^3(4) - (7)^2(30) - 7(30) - 30$

$= (7)^3(4) - \left[30 + (30)(7) + (7)^2(30)\right]$

$a = 30 \quad r = 7$

$$\Rightarrow u_n = (7)^n(4) - \frac{30\left(1 - (7)^n\right)}{1 - 7}$$

$$= (7)^n(4) + 5\left(1 - (7)^n\right)$$

$$= (4 - 5)(7)^n + 5$$

$$= \boxed{-1(7)^n + 5}$$

v) $u_n = -\frac{1}{2}u_{n-1} + 3$, $u_1 = 10$

$n=1$

$u_1 = -\frac{1}{2}u_0 + 3$

$10 = -\frac{1}{2}u_0 + 3$

$\frac{1}{2}u_0 = -7$

$u_0 = -14$

$\Rightarrow u_1 = -\frac{1}{2}(-14) + 3$

$n=2$

$u_2 = -\frac{1}{2}u_1 + 3$

$= -\frac{1}{2}\left(-\frac{1}{2}(-14) + 3\right) + 3$

$= \left(-\frac{1}{2}\right)^2(-14) - \frac{1}{2}(3) + 3$

$n=3$

$u_3 = -\frac{1}{2}u_2 + 3$

$= -\frac{1}{2}\left(\left(-\frac{1}{2}\right)^2(-14) - \frac{1}{2}(3) + 3\right) + 3$

$= \left(-\frac{1}{2}\right)^3(-14) + \left(\frac{1}{2}\right)^2(3) - \frac{1}{2}(3) + 3$

$a = 3 \quad r = -\frac{1}{2}$

$\Rightarrow u_n = \left(-\frac{1}{2}\right)^n(-14) + \frac{3\left(1 - \left(-\frac{1}{2}\right)^n\right)}{1 - \left(-\frac{1}{2}\right)}$

$= \left(-\frac{1}{2}\right)^n(-14) + 2\left(1 - \left(-\frac{1}{2}\right)^n\right)$

$= (-14 - 2)\left(-\frac{1}{2}\right)^n + 2$

$= \boxed{-16\left(-\frac{1}{2}\right)^n + 2}$

vi) $u_n = \frac{2}{3}u_{n-1} - 1$, $u_0 = 18$

$n=1$

$u_1 = \frac{2}{3}u_0 - 1$

$= \frac{2}{3}(18) - 1$

$n=2$

$u_2 = \frac{2}{3}u_1 - 1$

$= \frac{2}{3}\left(\frac{2}{3}(18) - 1\right) - 1$

$= \left(\frac{2}{3}\right)^2(18) - \frac{2}{3}(1) - 1$

$n=3$

$u_3 = \frac{2}{3}u_2 - 1$

$= \frac{2}{3}\left(\left(\frac{2}{3}\right)^2(18) - \frac{2}{3}(1) - 1\right) - 1$

$= \left(\frac{2}{3}\right)^3(18) - \left(\frac{2}{3}\right)^2(1) - \frac{2}{3} - 1$

$= \left(\frac{2}{3}\right)^3(18) - \left[1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2(1)\right]$

$a = 1 \quad r = \frac{2}{3}$

$\Rightarrow u_n = \left(\frac{2}{3}\right)^n(18) - \frac{1\left(1 - \left(\frac{2}{3}\right)^n\right)}{1 - \frac{2}{3}}$

$= \left(\frac{2}{3}\right)^n(18) - 3\left(1 - \left(\frac{2}{3}\right)^n\right)$

$= (18 + 3)\left(\frac{2}{3}\right)^n - 3$

$= \boxed{21\left(\frac{2}{3}\right)^n - 3}$

Q2.

i) $u_n = \frac{2}{3}u_{n-1} + 9 \qquad u_0 = 162$

$n=1$

$u_1 = \frac{2}{3}u_0 + 9$

$= \frac{2}{3}(162) + 9$

$n=2$

$u_2 = \frac{2}{3}u_1 + 9$

$= \frac{2}{3}\left(\frac{2}{3}(162) + 9\right) + 9$

$= \left(\frac{2}{3}\right)^2(162) + \frac{2}{3}(9) + 9$

$n=3$

$u_3 = \frac{2}{3}u_2 + 9$

$= \frac{2}{3}\left(\left(\frac{2}{3}\right)^2(162) + \frac{2}{3}(9) + 9\right) + 9$

$= \left(\frac{2}{3}\right)^3(162) + \underbrace{\left(\frac{2}{3}\right)^2(9) + \frac{2}{3}(9) + 9}_{a=9 \quad r=2/3}$

$\Rightarrow u_n = \left(\frac{2}{3}\right)^n(162) + \frac{9\left(1 - \left(\frac{2}{3}\right)^n\right)}{1 - \frac{2}{3}}$

$= \left(\frac{2}{3}\right)^n(162) + 27\left(1 - \left(\frac{2}{3}\right)^n\right)$

$= (162 - 27)\left(\frac{2}{3}\right)^n + 27$

$= \boxed{135\left(\frac{2}{3}\right)^n + 27}$

ii) $u_8 = 135\left(\frac{2}{3}\right)^8 + 27$

$= \boxed{\frac{7841}{243}}$

iii) $\lim_{n \to \infty} 135\left(\frac{2}{3}\right)^n + 27$

$= 135(0) + 27$

$= \boxed{27}$

Q3.

i) $u_n = \frac{1}{4}u_{n-1} + 90$, $u_1 = 800$

$n = 1$

$u_1 = \frac{1}{4}(u_0) + 90$

$800 = \frac{1}{4}(u_0) + 90$

$\frac{1}{4}u_0 = 710$

$u_0 = 2840$

$\Rightarrow u_1 = \frac{1}{4}(2840) + 90$

$n = 2$

$u_2 = \frac{1}{4}u_1 + 90$

$= \frac{1}{4}\left(\frac{1}{4}(2840) + 90\right) + 90$

$= \left(\frac{1}{4}\right)^2(2840) + \frac{1}{4}(90) + 90$

$n = 3$

$u_3 = \frac{1}{4}u_2 + 90$

$= \left(\frac{1}{4}\right)^3(2840) + \underbrace{\left(\frac{1}{4}\right)^2(90) + \frac{1}{4}(90) + 90}_{a = 90 \quad r = 1/4}$

$\Rightarrow u_n = \left(\frac{1}{4}\right)^n(2840) + \frac{90\left(1 - \left(\frac{1}{4}\right)^n\right)}{1 - 1/4}$

$= \left(\frac{1}{4}\right)^n(2840) + 120\left(1 - \left(\frac{1}{4}\right)^n\right)$

$= (2840 - 120)\left(\frac{1}{4}\right)^n + 120$

$= \boxed{2720\left(\frac{1}{4}\right)^n + 120}$

ii) 6th term starting @ $u_1 = u_6$

$u_6 = 2720\left(\frac{1}{4}\right)^6 + 120$

$= \boxed{\frac{15445}{128}}$ or $\boxed{120.664}$

iii) $2720\left(\frac{1}{4}\right)^n + 120 < 125$

$2720\left(\frac{1}{4}\right)^n < 5$

$\left(\frac{1}{4}\right)^n < \frac{5}{2720}$

$\log\left(\frac{1}{4}\right)^n < \log\left(\frac{5}{2720}\right)$

$n\log\left(\frac{1}{4}\right) < \log\left(\frac{5}{2720}\right)$

$\Rightarrow n > \frac{\log\left(5/2720\right)}{\log\left(1/4\right)}$ (as $\log\left(\frac{1}{4}\right) < 0$)

$n > 4.54 \Rightarrow \boxed{n = 5}$

OR Just work out terms:

$u_1 = 800$ $u_2 = 290$ $u_3 = 162.5$

$u_4 = 130.625$ $u_5 = 122.656$

iv) $\lim_{n \to \infty} 2720\left(\frac{1}{4}\right)^n + 120$

$= 2720(0) + 120 = \boxed{120}$

Q4.

(i) $u_{n+2} + 7u_{n+1} = -12u_n \quad u_0 = 2 \quad u_1 = -9$

$u_{n+2} + 7u_{n+1} + 12u_n = 0$

Char eqn:

$x^2 + 7x + 12 = 0$

$(x+4)(x+3) = 0$

$x+4 = 0 \quad \text{or} \quad x+3 = 0$

$x = -4 \qquad x = -3$

Theorem:

$u_n = l(-4)^n + m(-3)^n$

$u_0 = 2$

$l(-4)^0 + m(-3)^0 = 2$

$l + m = 2 \quad : \text{I}$

$u_1 = -9$

$l(-4)^1 + m(-3)^1 = -9$

$-4l - 3m = -9$

$4l + 3m = 9 \quad : \text{II}$

Solving I & II:

I $\times 3$: $3l + 3m = 6$

II: $(-)\ 4l + 3m = 9$

$-l = -3$

$l = 3$

Using I: $3 + m = 2$

$m = -1$

$\Rightarrow u_n = \boxed{3(-4)^n - 1(-3)^n}$

(ii) $2T_n - 5T_{n-1} = 12T_{n-2}$ $\quad u_1 = 1 \quad u_2 = 20$

$2T_n - 5T_{n-1} - 12T_{n-2} = 0$

Char eqn:

$2x^2 - 5x - 12 = 0$

$(2x + 3)(x - 4) = 0$

$2x + 3 = 0$ or $x - 4 = 0$

$2x = -3$ $\quad x = 4$

$x = -\frac{3}{2}$

Theorem:

$u_n = l\left(-\frac{3}{2}\right)^n + m(4)^n$

$u_1 = 1$

$l\left(-\frac{3}{2}\right)^1 + m(4)^1 = 1$

$-3l + 8m = 2 \quad : \text{I}$

$u_2 = 20$

$l\left(-\frac{3}{2}\right)^2 + m(4)^2 = 20$

$l\left(\frac{9}{4}\right) + 16m = 20$

$9l + 64m = 80 \quad : \text{II}$

Solving I & II:

I $\times 8$: $-24l + 64m = 16$

II : $(-)\ 9l \overset{(-)}{+} 64m \overset{(-)}{=} 80$

$-33l = -64$

$l = \frac{64}{33}$

Using I: $-3\left(\frac{64}{33}\right) + 8m = 2$

$-\frac{64}{11} + 8m = 2$

$8m = 2 + \frac{64}{11}$

$8m = \frac{86}{11}$

$m = \frac{43}{44}$

$\Rightarrow u_n = \boxed{\frac{64}{33}\left(-\frac{3}{2}\right)^n + \frac{43}{44}(4)^n}$

(iii) $2T_{n+1} + T_{n+2} = 15T_n$, $T_1 = 1$ $T_2 = 14$

$T_{n+2} + 2T_{n+1} - 15T_n = 0$

Char eqn:

$x^2 + 2x - 15 = 0$

$(x+5)(x-3) = 0$

$x+5 = 0$ or $x-3 = 0$

$x = -5$ $x = 3$

Theorem:

$T_n = l(-5)^n + m(3)^n$

$T_1 = 1$

$l(-5)^1 + m(3)^1 = 1$

$-5l + 3m = 1$: I

$T_2 = 14$

$l(-5)^2 + m(3)^2 = 14$

$25l + 9m = 14$: II

Solving I & II:

I × 3: $-15l + 9m = 3$

II: $(-)25l \overset{(-)}{+} 9m = {}^{(-)}14$

$-40l = -11$

$l = \frac{11}{40}$

Using I: $-5(\frac{11}{40}) + 3m = 1$

$-\frac{11}{8} + 3m = 1$

$3m = 1 + \frac{11}{8}$

$3m = \frac{19}{8}$

$m = \frac{19}{24}$

$\Rightarrow T_n = \boxed{\frac{11}{40}(-5)^n + \frac{19}{24}(3)^n}$

(iv) $u_{n+2} - 6u_{n+1} = -9u_n$, $u_0 = 3$ $u_1 = 6$

$u_{n+2} - 6u_{n+1} + 9u_n = 0$

Char eqn:

$x^2 - 6x + 9 = 0$

$(x-3)(x-3) = 0$

$x - 3 = 0$ or $x - 3 = 0$

$x = 3$ $x = 3$

Theorem:

$u_n = l(3)^n + mn(3)^n$

$u_0 = 2$

$l(3)^0 + m(0)(3)^0 = 3$

$l = 3$

$u_1 = 6$

$l(3)^1 + m(1)(3)^1 = 6$

$3l + 3m = 6$

$3(3) + 3m = 6$

$9 + 3m = 6$

$3m = -3$

$m = -1$

$\Rightarrow u_n = 3(3)^n - n(3)^n$

$= \boxed{(3-n)(3)^n}$

Q5.

i) $u_n - 3u_{n-1} = n - 5 \qquad u_0 = 1$

Complementary Soln:

Char eqn: $x^2 - 3x = 0$

$x(x-3) = 0$

$x = 0 \quad \text{or} \quad x - 3 = 0$

$x = 3$

$\Rightarrow u_n = L(0)^n + m(3)^n$

$= m(3)^n$

Particular Soln:

$n - 5$ is linear $\Rightarrow u_n = an + b$

$\Rightarrow u_{n-1} = a(n-1) + b$

$= an - a + b$

$u_n - 3u_{n-1} = n - 5$

$an + b - 3(an - a + b) = n - 5$

$an + b - 3an + 3a - 3b = n - 5$

$-2an + 3a - 2b = n - 5$

$\Rightarrow -2a = 1 \qquad 3a - 2b = -5$

$a = -\frac{1}{2} \qquad 3(-\frac{1}{2}) - 2b = -5$

$-\frac{3}{2} - 2b = -5$

$-3 - 6b = -10$

$-6b = -7$

$b = \frac{-7}{-6} = \frac{7}{6}$

$\Rightarrow$ Total Soln $= m(3)^n - \frac{1}{2}n + \frac{7}{6}$

$u_0 = 1$

$m(3)^0 - \frac{1}{2}(0) + \frac{7}{6} = 1$

$m + \frac{7}{6} = 1$

$m = 1 - \frac{7}{6} = -\frac{1}{6} \Rightarrow u_n = \boxed{-\frac{1}{6}(3)^n - \frac{1}{2}n + \frac{7}{6}}$

ii) $u_9 = -\frac{1}{6}(3)^9 - \frac{1}{2}(9) + \frac{7}{6} = \boxed{-\frac{19703}{6}}$ or $\boxed{-3283.83}$

Q6.

i) $u_n = 2n^2 - 1 - 2u_{n-1}$ $\quad u_1 = 5$

$u_n + 2u_{n-1} = 2n^2 - 1$

Complementary Soln:

Char eqn: $x^2 + 2x = 0$

$x(x+2) = 0$

$x = 0$ or $x + 2 = 0$

$x = -2$

$\Rightarrow u_n = l(0)^n + m(-2)^n$

$= m(-2)^n$

Particular Soln:

$2n^2 - 1$ is quadratic $\Rightarrow u_n = an^2 + bn + c$

$\Rightarrow u_{n-1} = a(n-1)^2 + b(n-1) + c$

$= a(n^2 - 2n + 1) + bn - b + c$

$= an^2 - 2an + a + bn - b + c$

$u_n + 2u_{n-1} = 2n^2 - 1$

$an^2 + bn + c + 2(an^2 - 2an + a + bn - b + c) = 2n^2 - 1$

$an^2 + bn + c + 2an^2 - 4an + 2a + 2bn - 2b + 2c = 2n^2 - 1$

$3an^2 + (3b - 4a)n + 3c + 2a - 2b = 2n^2 + 0n - 1$

$\Rightarrow 3a = 2$ $\quad 3b - 4a = 0$ $\quad 3c + 2a - 2b = -1$

$a = \frac{2}{3}$ $\quad 3b = 4a$ $\quad 3c = -1 - 2\left(\frac{2}{3}\right) + 2\left(\frac{8}{9}\right)$

$3b = 4\left(\frac{2}{3}\right) = \frac{8}{3}$ $\quad 3c = \frac{-5}{9}$

$b = \frac{8}{9}$ $\quad c = \frac{-5}{27}$

$\Rightarrow$ Total Soln $= m(-2)^n + \frac{2}{3}n^2 + \frac{8}{9}n - \frac{5}{27}$

$u_1 = 5$

$m(-2)^1 + \frac{2}{3}(1)^2 + \frac{8}{9}(1) - \frac{5}{27} = 5$

$-2m = \frac{98}{27}$

$m = -\frac{49}{27}$ $\Rightarrow u_n = \boxed{-\frac{49}{27}(-2)^n + \frac{2}{3}n^2 + \frac{8}{9}n - \frac{5}{27}}$

ii) $u_7 = -\frac{49}{27}(-2)^7 + \frac{2}{3}(7)^2 + \frac{8}{9}(7) - \frac{5}{27} = \boxed{271}$

Q7.

i) $T_n = 7T_{n-1} - 12T_{n-2} + 8 - \frac{1}{2}(2)^n \quad T_1 = 2 \quad T_2 = 3$

$T_n - 7T_{n-1} + 12T_{n-2} = 8 - \frac{1}{2}(2)^n$

Complementary Soln:

Char eqn: $x^2 - 7x + 12 = 0$

$(x-4)(x-3) = 0$

$x - 4 = 0 \quad x - 3 = 0$

$x = 4 \quad x = 3$

$\Rightarrow u_n = l(4)^n + m(3)^n$

Particular Soln:

$8 - \frac{1}{2}(2)^n$ is exponential $\Rightarrow u_n = a(2)^n + b$

$\Rightarrow u_{n-1} = a(2)^{n-1} + b \qquad u_{n-2} = a(2)^{n-2} + b$

$= \frac{a(2)^n}{2} + b \qquad = \frac{a(2)^n}{4} + b$

$T_n - 7T_{n-1} + 12T_{n-2} = 8 - \frac{1}{2}(2)^n$

$a(2)^n + b - 7\left(\frac{a}{2}(2)^n + b\right) + 12\left(\frac{a}{4}(2)^n + b\right) = 8 - \frac{1}{2}(2)^n$

$a(2)^n + b - \frac{7a}{2}(2)^n - 7b + 3a(2)^n + 12b = 8 - \frac{1}{2}(2)^n$

$\frac{a}{2}(2)^n + 6b = 8 - \frac{1}{2}(2)^n$

$\Rightarrow \frac{a}{2} = \frac{-1}{2} \qquad 6b = 8$

$a = -1 \qquad b = \frac{8}{6} = \frac{4}{3}$

$\Rightarrow$ Total Soln: $l(4)^n + m(3)^n - 1(2)^n + \frac{4}{3}$

$T_1 = 2$

$l(4)^1 + m(3)^1 - 1(2)^1 + \frac{4}{3} = 2$

$4l + 3m = \frac{8}{3} \quad : \text{I}$

$T_2 = 3$

$l(4)^2 + m(3)^2 - 1(2)^2 + \frac{4}{3} = 3$

$16l + 9m = \frac{17}{3} \quad : \text{II}$

Solving I & II:

$\text{I} \times 3: \quad 12l + 9m = 8$

$\text{II}: \quad (-)16l + (-)9m = (-)\frac{17}{3}$

$-4l = \frac{7}{3}$

$l = -\frac{7}{12}$

I: $4\left(\frac{-7}{12}\right) + 3m = \frac{8}{3}$

$3m = 5$

$m = \frac{5}{3}$

$\Rightarrow T_n = \boxed{-\frac{7}{12}(4)^n + \frac{5}{3}(3)^n - 2^n + \frac{4}{3}}$

ii) $T_{10} = -\frac{7}{12}(4)^{10} + \frac{5}{3}(3)^{10} - 2^{10} + \frac{4}{3} = \boxed{-514,277}$

Q8.

i) Birth rate = 1.6% Death rate = 1.2%

⇒ net change = 0.4%

Immigration = 40000 Emigration = 25000

⇒ net change = 15000

⇒ Popn in any year will be 1.4% of previous year and then 15000 will be added

⇒ $\boxed{u_n = 1.004\,u_{n-1} + 15000}$

ii) $n=1$

$u_1 = 1.004\,u_0 + 15000$

$= 1.004\,(5M) + 15000$

$n=2$

$u_2 = 1.004\,u_1 + 15000$

$= 1.004(1.004(5m) + 15000) + 15000$

$= (1.004)^2(5m) + 1.004(15000) + 15000$

$n=3$

$u_3 = 1.004\,u_2 + 15000$

$= 1.004\left((1.004)^2(5m) + 1.004(15000) + 15000\right) + 15000$

$= (1.004)^3(5m) + \underbrace{(1.004)^2(15000) + 1.004(15000) + 15000}_{a = 15000 \quad r = 1.004}$

⇒ $u_n = (1.004)^n(5{,}000{,}000) + \dfrac{15000\left(1 - (1.004)^n\right)}{1 - 1.004}$

$= (1.004)^n(5{,}000{,}000) - 3750000\left(1 - (1.004)^n\right)$

$= (1.004)^n(5{,}000{,}000 + 3{,}750{,}000) - 3{,}750{,}000$

$= \boxed{8{,}750{,}000\,(1.004)^n - 3{,}750{,}000}$

iii) $u_{30} = 8750000(1.004)^{30} - 3{,}750{,}000$

$= \boxed{6{,}113{,}236}$

Q9. Loan gains 1.2% interest each month

i) $\Rightarrow$ debt is 1.012 of previous month's debt Repays: €300

$\Rightarrow$ $\boxed{u_n = 1.012\,u_{n-1} - 300}$ Initial debt = $u_0 = 10{,}000$

ii) $n=1$

$u_1 = 1.012\,u_0 - 300$

$= 1.012(10000) - 300$

$n=2$

$u_2 = 1.012\,u_1 - 300$

$= 1.012(1.012(10000) - 300) - 300$

$= (1.012)^2(10000) - 1.012(300) - 300$

$n=3$

$u_3 = 1.012\,u_2 - 300$

$= (1.012)^3(10000) - (1.012)^2(300) - 1.012(300) - 300$

$= (1.012)^3(10000) - [300 + 300(1.012) + 300(1.012)^2]$

$a = 300 \quad r = 1.012$

$\Rightarrow u_n = (1.012)^n(10000) - \frac{300(1-(1.012)^n)}{1-1.012}$

$= (1.012)^n(10000) + 25000(1-(1.012)^n)$

$= (1.012)^n(10000 - 25000) + 25000$

$= \boxed{25000 - 15000(1.012)^n}$

iii) Debt = 0 $\Rightarrow$ $25000 - 15000(1.012)^n = 0$

$15000(1.012)^n = 25000$

$(1.012)^n = \frac{25000}{15000} = \frac{5}{3}$

$\log(1.012)^n = \log\left(\frac{5}{3}\right)$

$n\log(1.012) = \log\left(\frac{5}{3}\right)$

$n = \frac{\log(5/3)}{\log(1.012)} = 42.8 = \boxed{43 \text{ months}}$

iv) $u_{42} = 25000 - 15000(1.012)^{42}$

$= €244.438$

$\Rightarrow$ Final payment $= 244.438 \times 1.012$

$= \boxed{€247.37}$

Q10.

i) $D_n = 1.003\,D_{n-1} - A$ $\qquad$ 35 yrs = 420 months

ii) $n = 1$

$D_1 = 1.003\,D_0 - A$

$= 1.003(300{,}000) - A$

$n = 2$

$D_2 = 1.003\,D_1 - A$

$= 1.003(1.003(300000) - A) - A$

$= (1.003)^2(300000) - 1.003A - A$

$n = 3$

$D_3 = 1.003\,D_2 - A$

$= 1.003((1.003)^2(300{,}000) - 1.003A - A) - A$

$= (1.003)^3(300{,}000) - \underbrace{[A + 1.003A + (1.003)^2A]}_{a = A \quad r = 1.003}$

$$\Rightarrow D_n = (1.003)^n(300{,}000) - \frac{A(1 - (1.003)^n)}{1 - 1.003}$$

$$= (1.003)^n(300{,}000) + \frac{1000A}{3}(1 - (1.003)^n)$$

$D_n = 0$ @ $n = 420$

$$(1.003)^{420}(300{,}000) + \frac{1000A}{3}(1 - (1.003)^{420}) = 0$$

$$\frac{1000A}{3}(1 - (1.003)^{420}) = -300000(1.003)^{420}$$

$$\frac{1000A}{3} = \frac{-300{,}000(1.003)^{420}}{1 - (1.003)^{420}}$$

$$\frac{1000A}{3} = 419{,}105.3755$$

$$1000A = 1257316.126$$

$$A = \boxed{€1257.32}$$

Q11. Try with smaller figures i.e. 2 pairs to start

i)

$n = 0$ A B 2 pairs

$n = 1$ A B 2 pairs

$n = 2$ A B C D E F } $2 + 4 = 6$ pairs

$n = 3$ A B G H I J C, D, E, F } $2 + 4 + 4 = 10$ pairs

$n = 4$ A B C D E F K L M N O P Q R S T U V , G, H, I, F } 22 pairs

Sequence: 2, 2, 6, 10, 22 .....

$6 = 2 + 2(2)$ $\quad 10 = 6 + 2(2)$ $\quad 22 = 10 + 2(6)$

$\Rightarrow \boxed{R_n = R_{n-1} + 2R_{n-2}}$

ii) Char eqn: $x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0$

$x = +2 \quad x = -1$

Theorem: $R_n = l(2)^n + m(-1)^n$

$R_0 = 8$

$l(2)^0 + m(-1)^0 = 8$

$l + m = 8 \quad : \text{I}$

$R_1 = 8$

$l(2)^1 + m(-1)^1 = 8$

$2l - m = 8 \quad : \text{II}$

Solving I & II:

I: $l + m = 8$

II: $2l - m = 8$

$3l = 16$

$l = 16/3$

I: $\frac{16}{3} + m = 8$

$m = \frac{8}{3}$

$\Rightarrow R_n = \boxed{\frac{16}{3}(2)^n + \frac{8}{3}(-1)^n}$

$R_5 = \frac{16}{3}(2)^5 + \frac{8}{3}(-1)^5$

$= \boxed{168}$

Q12. $P_n = \frac{1}{4}(3P_{n-1} + P_{n-2})$

i) $P_n - \frac{3}{4}P_{n-1} - \frac{1}{4}P_{n-2} = 0$

$4P_n - 3P_{n-1} - P_{n-2} = 0$

Char Eqn:

$4x^2 - 3x - 1 = 0$

$(4x + 1)(x - 1) = 0$

$4x + 1 = 0$ or $x - 1 = 0$

$4x = -1$ $\quad x = 1$

$x = -\frac{1}{4}$

$\Rightarrow$ Theorem: $P_n = l\left(-\frac{1}{4}\right)^n + m(1)^n$

$= l\left(-\frac{1}{4}\right)^n + m$

$P_0 = 70000$ $\quad P_1 = 95000$

$l\left(-\frac{1}{4}\right)^0 + m = 70000$ $\quad l\left(-\frac{1}{4}\right)^1 + m = 95000$

$l + m = 70000$: I $\quad -l + 4m = 380000$: II

Solving I & II:

I: $l + m = 70000$

II: $-l + 4m = 380000$

$5m = 450000$

$m = 90000$

I: $l + 90000 = 70000$

$l = -20000$

$\Rightarrow P_n = \boxed{90000 - 20000\left(-\frac{1}{4}\right)^n}$

ii) $2030 \Rightarrow n = 5$

$P_5 = 90000 - 20000\left(-\frac{1}{4}\right)^5$

$= 90019.53$

$= \boxed{90020}$

iii) $\lim_{n\to\infty}\left(90000 - 20000\left(-\frac{1}{4}\right)^n\right) = 90000 - 0 = \boxed{90000}$

Q13.

i) $C_n = C_{n-1} + 0.3C_{n-2} + 5n + 40 \qquad C_1 = 100 \quad C_2 = 175$

$C_3 = 175 + 0.3(100) + 5(3) + 40$

$= \boxed{260}$

$C_4 = 260 + 0.3(175) + 5(4) + 40$

$= 372.5 = \boxed{373}$

ii) Comp Soln:

$C_n - C_{n-1} - 0.3C_{n-2} = 0$

$x^2 - x - 0.3 = 0$

$x = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(0.3)}}{2(1)}$

$= \dfrac{5 \pm \sqrt{55}}{10}$

$\Rightarrow \quad C_n = l\left(\frac{5+\sqrt{55}}{10}\right)^n + m\left(\frac{5-\sqrt{55}}{10}\right)^n$

Particular Soln:

$5n + 40$ is linear $\Rightarrow \quad C_n = an + b$

$C_{n-1} = a(n-1) + b = an - a + b$

$C_{n-2} = a(n-2) + b = an - 2a + b$

$C_n - C_{n-1} - 0.3C_{n-2} = 5n + 40$

$an + b - (an - a + b) - 0.3(an - 2a + b) = 5n + 40$

$an + b - an + a - b - 0.3an + 0.6a - 0.3b = 5n + 40$

$-0.3an + 1.6a - 0.3b = 5n + 40$

$-0.3a = 5 \qquad 1.6a - 0.3b = 40$

$a = -\frac{50}{3} \qquad 1.6\left(-\frac{50}{3}\right) - 0.3b = 40$

$-0.3b = \frac{200}{3}$

$b = -\frac{2000}{9}$

$\Rightarrow$ Total Soln: $C_n = l\left(\frac{5+\sqrt{55}}{10}\right)^n + m\left(\frac{5-\sqrt{55}}{10}\right)^n - \frac{50n}{3} - \frac{2000}{9}$

$C_1 = 100$

$$l\left(\frac{5+\sqrt{55}}{10}\right) + m\left(\frac{5-\sqrt{55}}{10}\right) - \frac{50(1)}{3} - \frac{2000}{9} = 100$$

$$9l(5+\sqrt{55}) + 9m(5-\sqrt{55}) - 1500 - 20000 = 9000 \quad (\times 90)$$

$$9l(5+\sqrt{55}) + 9m(5-\sqrt{55}) = 30500 \quad : \text{I}$$

$C_2 = 175$

$$l\left(\frac{5+\sqrt{55}}{10}\right)^2 + m\left(\frac{5-\sqrt{55}}{10}\right)^2 = \frac{50(2)}{3} - \frac{2000}{9} = 175$$

$$l\left(\frac{8+\sqrt{55}}{10}\right) + m\left(\frac{8-\sqrt{55}}{10}\right) - \frac{100}{3} - \frac{2000}{9} = 175$$

$$9l(8+\sqrt{55}) + 9m(8-\sqrt{55}) - 3000 - 20000 = 15750$$

$$9l(8+\sqrt{55}) + 9m(8-\sqrt{55}) = 38750 \quad : \text{II}$$

Solving I & II:

I: $111.75l - 21.75m = 30500$

II: $138.75l + 5.25m = 38750$

I $\times 5.25$: $586.6875l - 114.1875m = 160125$

II $\times 21.75$: $3017.8125l + 114.1875m = 842812.5$

$$3604.5l = 1002937.5$$

$$l = 278.25$$

$\Rightarrow$ I: $111.75(278.25) - 21.75m = 30500$

$$31094.4375 - 21.75m = 30500$$

$$-21.75m = -594.4375$$

$$m = 27.33$$

$$\Rightarrow C_n = \boxed{278.25\left(\frac{5+\sqrt{55}}{10}\right)^n + 27.33\left(\frac{5-\sqrt{55}}{10}\right)^n - \frac{50n}{3} - \frac{2000}{9}}$$

$$C_{10} = 278.25\left(\frac{5+\sqrt{55}}{10}\right)^{10} + 27.33\left(\frac{5-\sqrt{55}}{10}\right)^{10} - \frac{50(10)}{3} - \frac{2000}{9}$$

$$= 2033.93$$

$$= \boxed{2034}$$

[2023 HL Sample Paper Q8]

n = 0

=> $P_1 = 1.03P_0 - B$

=> $P_1 = 1.03(8000) - B$

n = 1

=> $P_2 = 1.03P_1 - B$

=> $P_2 = 1.03(1.03(8000) - B) - B$

=> $P_2 = 1.03^2(8000) - 1.03B - B$

n = 2

=> $P_3 = 1.03P_2 - B$

=> $P_3 = 1.03(1.03^2(8000) - 1.03B - B) - B$

=> $P_3 = 1.03^3(8000) - 1.03^2B - 1.03B - B$

Looking at P_1, P_2, P_3 above:

$$P_n = 1.03^n(8000) - [B + B(1.03) + B(1.03)^2 + \cdots]$$

The terms in the square brackets form a Geometric Series with a = B and r = 1.03

$$\Rightarrow P_n = 1.03^n(8000) - \left[\frac{B(1 - 1.03^n)}{1 - 1.03}\right] \quad \text{(Using the } Sn \text{ formula from Tables)}$$

$$\Rightarrow P_n = 1.03^n(8000) + \frac{100}{3}[B(1 - 1.03^n)]$$

$$\Rightarrow P_n = 1.03^n(8000) + \frac{100}{3}B - \frac{100}{3}B(1.03^n)$$

$$\Rightarrow P_n = 1.03^n\left(8000 - \frac{100}{3}B\right) + \frac{100}{3}B$$

$$\Rightarrow P_n = 1.03^n\left(\frac{24000 - 100B}{3}\right) + \frac{100}{3}B$$

2024 HL Q9

(i)

$M_2 = M_1 + \frac{3M_0}{4} = 245 + \frac{3(200)}{4} = 395$	$M_3 = M_2 + \frac{3M_1}{4} = 395 + \frac{3(245)}{4} = 578.75$

(ii)

$M_{n+2} = M_{n+1} + \frac{3M_n}{4}$

=> $4M_{n+2} - 4M_{n+1} - 3M_n = 0$

=> Characteristic Equation = $4x^2 - 4x - 3 = 0$

$(2x - 3)(2x + 1) = 0$

=> $2x - 3 = 0$ or $2x + 1 = 0$

=> $x = \frac{3}{2}$ or $x = -\frac{1}{2}$

Theorem: $M_n = l(\frac{3}{2})^n + m(-\frac{1}{2})^n$

$\underline{M_0 = 200}$ $l(\frac{3}{2})^0 + m\left(-\frac{1}{2}\right)^0 = 200$ => $l + m = 200$ => $m = 200 - l$ .....Eqn 1	$\underline{M_1 = 245}$ $l(\frac{3}{2})^1 + m\left(-\frac{1}{2}\right)^1 = 245$ => $3l - m = 490$ Subbing in Eqn 1: => $3l - (200 - l) = 490$ => $4l - 200 = 490$ => $4l = 690$ => $l = \frac{690}{4} = 172.5$ => $m = 200 - 172.5 = 27.5$

=> $M_n = 172.5(\frac{3}{2})^n + 27.5(-\frac{1}{2})^n$

(iii)

$M_0 = 200, M_1 = 245, M_2 = 395, M_3 = 578.75$

$M_4 = 172.5\left(\frac{3}{2}\right)^4 + 27.5\left(-\frac{1}{2}\right)^4 = 875$	$M_5 = 172.5(\frac{3}{2})^5 + 27.5(-\frac{1}{2})^5 = 1309.0625$

Total $= M_0 + M_1 + M_2 + M_3 + M_4 + M_5 = 3602.8125\ kg$

(iv)

$$P_{n+2} = P_{n+1} + \frac{3P_n}{4} - 2^{n+2}$$

=> Equation 1: $4P_{n+2} - 4P_{n+1} - 3P_n = -4.2^{n+2}$ (Multiplying through by 4)

The characteristic equation will be the same as the part (ii), so the complimentary solution will be: $P_n = 172.5(\frac{3}{2})^n + 27.5(-\frac{1}{2})^n$

As the right-hand side is an exponential, Particular solution would be in the form:

$$P_n = k.2^{n+2}$$

=> $P_{n+2} = k.2^{(n+2)+2} = k.2^{n+4}$ and $P_{n+1} = k.2^{(n+2)+1} = k.2^{n+3}$

Subbing into Equation 1 above gives:

$$4k.2^{n+4} - 4k.2^{n+3} - 3k.2^{n+2} = -4.2^{n+2}$$

Breaking up all powers of 2 into the form 2^{n+2} gives:

$$4k.2^{n+2}.2^2 - 4k.2^{n+2}.2^1 - 3k.2^{n+2} = -4.2^{n+2}$$

$$\Rightarrow 16k.2^{n+2} - 8k.2^{n+2} - 3k.2^{n+2} = -4.2^{n+2}$$

$$\Rightarrow 5k.2^{n+2} = -4.2^{n+2}$$

$$\Rightarrow 5k = -4$$

$$\Rightarrow k = -\frac{4}{5}$$

=> Total solution will be: $P_n = l(\frac{3}{2})^n + m(-\frac{1}{2})^n - \frac{4}{5}(2)^{n+2}$

$\underline{P_0 = 199}$	$\underline{P_1 = 243}$
$l(\frac{3}{2})^0 + m(-\frac{1}{2})^0 - \frac{4}{5}(2)^{0+2} = 199$	$l(\frac{3}{2})^0 + m(-\frac{1}{2})^0 - \frac{4}{5}(2)^{0+2} = 243$
$\Rightarrow l + m - \frac{16}{5} = 199$	$\Rightarrow \frac{3l}{2} - \frac{m}{2} - \frac{32}{5} = 243$
$\Rightarrow 5l + 5m - 16 = 995$	$\Rightarrow 15l - 5m = 2494$
$\Rightarrow 5l + 5m = 1011$	$\Rightarrow 15l - (1011 - 5l) = 2494$ (Using Eqn 1)
$\Rightarrow 5m = 1011 - 5l$ .....Eqn 1	$\Rightarrow 20l = 2494 + 1011$
	$\Rightarrow l = \frac{3505}{20} = \frac{701}{4} = 175.25$
	$\Rightarrow m = \frac{539}{20} = 26.95$

$$\Rightarrow P_n = 175.25(\frac{3}{2})^n + 26.95(-\frac{1}{2})^n - \frac{4}{5}(2)^{n+2}$$

(v)

$$P_0 = 199, P_1 = 243$$

$P_2 = 175.25\left(\frac{3}{2}\right)^2 + 26.95\left(-\frac{1}{2}\right)^2 - \frac{4}{5}(2)^{2+2} = 388.25$	$P_3 = 562.5, P_4 = 837.6875$ $P_5 = 1227.5625$

Total $= P_0 + P_1 + P_2 + P_3 + P_4 + P_5 = 3458\ kg$