Differentiation

Q1. $f(x) = 2x^2 - 3x + 1$

$f(x+h) = 2(x+h)^2 - 3(x+h) + 1$

$= 2(x^2 + 2hx + h^2) - 3x - 3h + 1$

$= 2x^2 + 4hx + 2h^2 - 3x - 3h + 1$

$f(x+h) - f(x) =$

$2x^2 + 4hx + 2h^2 - 3x - 3h + 1 - (2x^2 - 3x + 1)$

$\cancel{2x^2} + 4hx + 2h^2 - \cancel{3x} - 3h + \cancel{1} - \cancel{2x^2} + \cancel{3x} - \cancel{1}$

$= 4hx + 2h^2 - 3h$

$$\frac{f(x+h) - f(x)}{h} = \frac{4hx + 2h^2 - 3h}{h}$$

$$= 4x + 2h - 3$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 4x + 2(0) - 3$$

$$= \boxed{4x - 3}$$

Q2. $y = \sqrt{4 - 3x}$

$y = (4 - 3x)^{1/2}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}(4 - 3x)^{-\frac{1}{2}}(-3)$$

$$= \boxed{\frac{-3}{2\sqrt{4 - 3x}}}$$

Q3. $y = \ln\left(\frac{3+x}{\sqrt{9-x^2}}\right)$

$$\frac{dy}{dx} = \frac{1}{\frac{3+x}{\sqrt{9-x^2}}}\left[\frac{(\sqrt{9-x^2})(1) - (x+3)(\frac{1}{2})(9-x^2)^{-\frac{1}{2}}(-2x)}{(\sqrt{9-x^2})^2}\right]$$

$$= \frac{\sqrt{9-x^2}}{x+3}\left[\frac{\sqrt{9-x^2} + \frac{(x+3)(x)}{\sqrt{9-x^2}}}{9-x^2}\right]$$

$$= \frac{\sqrt{9-x^2}}{x+3}\left[\frac{\frac{9-x^2+x^2+3x}{\sqrt{9-x^2}}}{9-x^2}\right]$$

$$= \frac{\sqrt{9-x^2}}{x+3}\left[\frac{3x+9}{(9-x^2)\sqrt{9-x^2}}\right]$$

$$= \frac{3(\cancel{x+3})}{(\cancel{x+3})(9-x^2)}$$

$$= \boxed{\frac{3}{9-x^2}}$$

Q4 $y = \tan^{-1} \frac{5x}{1}$

i)
$$\frac{dy}{dx} = \frac{1}{(1)^2+(5x)^2} \cdot 5$$
$$= \frac{5}{1+25x^2}$$

@ $x=5$
$$\frac{dy}{dx} = \frac{5}{1+25\left(\frac{1}{\cdot}\right)^2}$$
$$= \boxed{4}$$

ii)
$$\sqrt{\frac{1+3x}{x-1}} = y$$

Log   Quotient Rule
$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1+3x}{x-1}\right)^{-\frac{1}{2}} \cdot \left[\frac{(x-1)(3)-(1+3x)(1)}{(x-1)^2}\right]$$
$$= \frac{1}{2}\sqrt{\frac{x-1}{3x+1}}\left[\frac{\cancel{3x}-3-1-\cancel{3x}}{(x-1)^2}\right]$$
$$= \frac{1}{2}\sqrt{\frac{x-1}{3x+1}}\left(\frac{-4}{(x-1)^2}\right)$$

@ $x=5$
$$\frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{4}{16}}\left(\frac{-4}{(4)^2}\right)$$
$$= \boxed{-\frac{1}{16}}$$

iii)
$$y = \frac{e^{3x}}{x} \quad \begin{matrix} u \\ v \end{matrix}$$
$$\frac{dy}{dx} = \frac{x(e^{3x}.3) - e^{3x}(1)}{x^2}$$
$$= \frac{3x.e^{3x} - e^{3x}}{x^2}$$
$$= \boxed{\frac{e^{3x}(3x-1)}{x^2}}$$

iv) $y = \ln(x^3+1)^3$
$$\frac{dy}{dx} = \underbrace{\frac{1}{(x^3+1)^3}}_{\text{Log}} \cdot \underbrace{3\cancel{(x^3+1)^2}}_{\text{Power}} \cdot \underbrace{(3x^2)}_{x^3+1}$$
$$= \boxed{\frac{9x^2}{x^3+1}}$$

v) $y = x^3.e^x$ (u)(v)
$$\frac{dy}{dx} = x^3.e^x + e^x(3x^2)$$
$$= \boxed{e^x(x^3+3x^2)}$$

vi) $y = e^{\cos x}$
$$\frac{dy}{dx} = e^{\cos x}(-\sin x)$$
$$= -\sin x . e^{\cos x}$$

@ $x=0$
$$\frac{dy}{dx} = -\sin(0).e^{\cos(0)}$$
$$= -0.e^1$$
$$= \boxed{0}$$

vii) $y = \cos^3(5x^2)$ P.T.A
$$\frac{dy}{dx} = 3\cos^2(5x^2).(-\sin(5x^2)).10x$$
$$= \boxed{-30x\cos^2(5x^2).\sin(5x^2)}$$

Q5. $f(x) = x^3 + kx^2 - 4$

$f'(x) = 3x^2 + 2kx$

@ local min/max $f'(x) = 0$

$\Rightarrow 3x^2 + 2kx = 0$

$x(3x + 2k) = 0$

$x = 0$ or $3x + 2k = 0$

$3x = -2k$

$x = -2k/3$

<u>To find y values</u>

@ $x = 0$ $y = (0)^3 + k(0)^2 - 4$

$= -4$

$\Rightarrow \boxed{(0, -4)}$

@ $x = -2k/3$

$y = \left(\frac{-2k}{3}\right)^3 + k\left(\frac{-2k}{3}\right)^2 - 4$

$= \frac{-8k^3}{27} + \frac{4k^3}{9} - 4$

$= \frac{-8k^3}{27} + \frac{12k^3}{27} - \frac{108}{27}$

$= \frac{4k^3 - 108}{27}$

$\Rightarrow \boxed{\left(-\frac{2k}{3}, \frac{4k^3 - 108}{27}\right)}$

Q6. $y = (x^2 - 2)e^{-2x}$ Product Rule

$\frac{dy}{dx} = (x^2 - 2).e^{-2x}(-2) + e^{-2x}(2x)$

$= e^{-2x}(4 - 2x^2 + 2x) = 0$

$\Rightarrow e^{-2x} = 0$ or $4 - 2x^2 + 2x = 0$

$2x^2 - 2x - 4 = 0$

$x^2 - x - 2 = 0$

$(x + 1)(x - 2) = 0$

$x = -1$ $x = 2$

@ $x = -1$ $y = ((-1)^2 - 2)e^{-2(-1)} = -e^2$

@ $x = 2$ $y = ((2)^2 - 2)e^{-2(2)} = 2/e^4$

$\Rightarrow$ Turning Pts: $\boxed{(-1, -e^2)\ (2, 2/e^4)}$

Q7. $f(x) = \frac{3}{x-2} = 3(x-2)^{-1}$

i) $f'(x) = -3(x-2)^{-2}(1)$ Chain Rule

$= \frac{-3}{(x-2)^2}$

As this is < 0 $\forall x \in R$

$\Rightarrow f(x)$ always decreasing

ii) Tangent $= 3x + 4y + k = 0$

Slope of Tgt $= \frac{-x}{y} = \frac{-3}{4}$

$\Rightarrow dy/dx = -3/4$

$\Rightarrow \frac{-3}{(x-2)^2} = \frac{-3}{4}$

$\Rightarrow (x-2)^2 = 4$

$\Rightarrow x^2 - 4x + 4 = 4$

$\Rightarrow x^2 - 4x = 0$

$x(x - 4) = 0$

$x = 0$ or $x = 4$

$\Rightarrow y = \frac{3}{0-2}$ or $y = \frac{3}{4-2}$

$= -3/2$ $= \frac{3}{2}$

These points must be on the tangent

$\Rightarrow 3(0) + 4(-\frac{3}{2}) + k = 0$

$\Rightarrow \boxed{k = 6}$

and $3(4) + 4(\frac{3}{2}) + k = 0$

$\boxed{k = -18}$

Q8. Given: $\frac{dV}{dt} = \frac{\pi}{10}$ cm^3/s

Want: $\frac{dr}{dt}$

Link: $V = \pi r^2 h$

$V = \pi r^2 (0.1)$

$V = 0.1\pi r^2$

Diff both sides w.r.t t:

$$\frac{dV}{dt} = 0.1\pi . 2r . \frac{dr}{dt}$$

$$\Rightarrow \frac{\cancel{\pi}}{10} = \cancel{0.1}\,\cancel{\pi}\,\cancel{2}(\cancel{5}) . \frac{dr}{dt}$$

$$\Rightarrow \frac{1}{10} = \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \boxed{0.1 \text{ cm/s}}$$

Q9. $s = 12 + 24t - 3t^2$

Speed $= \frac{ds}{dt} = 24 - 6t$

@ $t = 3$

Speed $= 24 - 6(3)$

$= \boxed{6 \text{ m/s}}$

Q10. $5x^2 + 5y^2 = 13$

Differentiate each term w.r.t. x:

$$10x + 10y . \frac{dy}{dx} = 0$$

$$10y . \frac{dy}{dx} = -10x$$

$$\frac{dy}{dx} = \frac{-10x}{10y} = \frac{-x}{y}$$

@ $(3,1) \Rightarrow dy/dx = -3/1 = -3$

$\Rightarrow$ Eqn: $y - 1 = -3(x-3)$

$y - 1 = -3x + 9$

$\boxed{3x + y - 10 = 0}$

Q11.

i) Given: $\frac{dr}{dt} = 0.25$ cm/s

Want: $\frac{dV}{dt}$

Link: $V = \pi r^2 h$

$\Rightarrow V = \pi r^2 (2r)$

$\Rightarrow V = 2\pi r^3$

Diff w.r.t. t:

$$\Rightarrow \frac{dV}{dt} = 2\pi(3r^2) . \frac{dr}{dt}$$

$$= 6\pi r^2 . \frac{dr}{dt}$$

$$\Rightarrow \frac{dV}{dt} = 6\pi(2)^2(0.25)$$

$$= \boxed{6\pi \text{ cm}^3/\text{s}}$$

ii) Can't link V and A directly so need to find $\frac{dr}{dt}$ first:

Given: $\frac{dV}{dt} = 5\pi$ cm^3/s

Want: $\frac{dr}{dt}$

Link: $V = 2\pi r^3$ from above

Diff w.r.t t:

$$\frac{dV}{dt} = 6\pi r^2 . \frac{dr}{dt}$$

$$5\cancel{\pi} = 6\cancel{\pi}(5)^2 . \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{5}{150} = \frac{1}{30} \text{ cm/s}$$

Now:

Want: $\frac{dA}{dt}$

Link: $A = 2\pi r h + 2\pi r^2$

$= 2\pi r(2r) + 2\pi r^2$

$= 4\pi r^2 + 2\pi r^2$

$\Rightarrow A = 6\pi r^2$

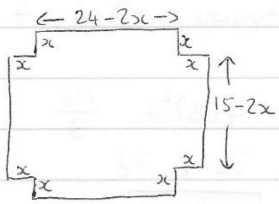
Diff w.r.t t:

$$\frac{dA}{dt} = 6\pi(2r) \cdot \frac{dr}{dt}$$

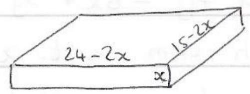
$$= 12\pi(5) \cdot \left(\frac{1}{30}\right)$$

$$= \boxed{2\pi \text{ cm}^2/\text{s}}$$

Q12.



When box folded up:



i). $V = L \times w \times h$

$= (24-2x)(15-2x)(x)$

$= (24-2x)(15x-2x^2)$

$= 24(15x-2x^2) - 2x(15x-2x^2)$

$= 360x - 48x^2 - 30x^2 + 4x^3$

$= 4x^3 - 78x^2 + 360x$

ii) Max $\Rightarrow \frac{dV}{dx} = 0$

$\frac{dV}{dx} = 12x^2 - 156x + 360 = 0$

$\Rightarrow x^2 - 13x + 30 = 0$

$(x-10)(x-3) = 0$

$x = 3$ or $x = 10$

x cannot be 10 as width would be negative $\Rightarrow \boxed{x=3}$

$\Rightarrow$ Vol max $= 4(3)^3 - 78(3)^2 + 360(3)$

$= \boxed{486 \text{ cm}^3}$

Q13. $y = \frac{x}{x-3}$ $\left(= \frac{u}{v}\right)$

Turning Points $\Rightarrow \frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{(x-3)(1) - (x)(1)}{(x-3)^2}$$

$$= \frac{x-3-x}{(x-3)^2}$$

$$= \frac{-3}{(x-3)^2}$$ which can never be 0

$\Rightarrow$ no turning points

Points of Inflection $\Rightarrow \frac{d^2y}{dx^2} = 0$

$$\frac{dy}{dx} = \frac{-3}{(x-3)^2} = -3(x-3)^{-2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = 6(x-3)^{-3}(1)$$ ← Chain Rule

$$= \frac{6}{(x-3)^3}$$ which can never be 0

$\Rightarrow$ no pts of inflection

Q14. Given: $\frac{dr}{dt} = 2$ cm/s

Want: $\frac{dV}{dt}$

Link: $V = \frac{4}{3}\pi r^3$

Diff w.r.t t:

$$\frac{dV}{dt} = \frac{4}{3}\pi(3r^2) \cdot \frac{dr}{dt}$$

$$= 4\pi r^2 \cdot \frac{dr}{dt}$$

$$\Rightarrow \frac{dV}{dt} = 4\pi(0.5)^2(2)$$

$$= \boxed{2\pi \text{ cm}^3/\text{s}}$$

Q15. $y = \sqrt{x^2 - 3}$

$y = (x^2-3)^{1/2}$

$\Rightarrow dy/dx = \frac{1}{\cancel{2}}(x^2-3)^{-\frac{1}{2}}(\cancel{2}x)$ ← Chain Rule

$= \dfrac{x}{\sqrt{x^2-3}}$

@ point (2,1)

$\dfrac{dy}{dx} = \dfrac{2}{\sqrt{(2)^2-3}}$

$= \dfrac{2}{\sqrt{1}} = \boxed{2}$

Eqn: $y - 1 = 2(x-2)$

$y - 1 = 2x - 4$

$\boxed{2x - y - 3 = 0}$

Q16. First need Area in terms of x

$V = L \times W \times h$

$\Rightarrow 72 = x(2x)(y)$

$\Rightarrow y = \dfrac{72}{2x^2}$

$\Rightarrow TSA = 2LW + 2Wh + 2Lh$

$= 2(x)(2x) + 2(\cancel{2x})\left(\dfrac{72}{\cancel{2}x^2}\right) + \cancel{2}(x)\left(\dfrac{72}{\cancel{2}x^2}\right)$

$= 4x^2 + \dfrac{144}{x} + \dfrac{72}{x}$

$= 4x^2 + \dfrac{216}{x}$

Min $\Rightarrow \dfrac{dA}{dx} = 0$

$\dfrac{dA}{dx} = 8x + 216(-1)(x)^{-2}$

$= 8x - \dfrac{216}{x^2} = 0$

$\Rightarrow 8x^3 - 216 = 0$

$8x^3 = 216$

$x^3 = 27$

$x = \sqrt[3]{27}$

$x = 3$

$\Rightarrow y = \dfrac{72}{2(3)^2} = 4$

$\Rightarrow$ Dimensions = 3, 6, 4

$\Rightarrow TSA = 4(3)^2 + \dfrac{216}{3}$

$= 36 + 72$

$= \boxed{108 \text{ cm}^2}$

Q17. $x^2 + y^2 - 6x + 3y + 8 = 0$

Diff each term w.r.t. x:

$2x + 2y \cdot \dfrac{dy}{dx} - 6 + 3 \cdot \dfrac{dy}{dx} + 0 = 0$

$2y \cdot \dfrac{dy}{dx} + 3 \cdot \dfrac{dy}{dx} = 6 - 2x$

$\dfrac{dy}{dx}(2y + 3) = 6 - 2x$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{6-2x}{2y+3}$.

@ (−1,2)

$\dfrac{dy}{dx} = \dfrac{6-2(-1)}{2(2)+3} = \boxed{\dfrac{8}{7}}$

Q18. $y = 4x^3 - 6x^2$

$\Rightarrow \dfrac{dy}{dx} = 12x^2 - 12x \qquad \dfrac{d^2y}{dx^2} = 24x - 12$

$x^2 \cdot \dfrac{d^2y}{dx^2} - 2x \cdot \dfrac{dy}{dx} - 12x^2 = 0$

$\Rightarrow x^2(24x - 12) - 2x(12x^2 - 12x) - 12x^2 = 0$

$\cancel{24x^3} - \cancel{12x^2} - \cancel{24x^3} + \cancel{24x^2} - \cancel{12x^2} = 0$

$0 = 0$

Q.E.D.