- Chapter 7: Motion In A Circle
- Topic 29: Radian Measure

  - Another unit of measurement that can be used for measuring angles instead of degrees is radians.
  - To convert between them, we need to know:

π radians = 180°

Show the measure of 1 radian in degrees on the board.

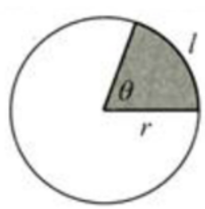
- Examples: i) Convert 145° to radians and ii) Convert $\frac{3\pi}{4}$ to degrees

Solution:

i) We want radians so write the conversion above with the radians on the right i.e.	ii) We want degrees so just fill in 180° instead of π i.e.
180° = π radians 1° = $\frac{\pi}{180}$ $(\div 180)$ 145° = $\frac{\pi}{180}$ x 145 $(\times\ 145)$ = $\frac{29\pi}{36}$ radians	$\frac{3\pi}{4} = \frac{3(180)}{4} = 135°$

Sector Formulae:

- In J.C. we learned how to find the area and length of a sector using the formulae on the left in the table below.
- If the angle in the sector is in radians, then we can use the formulae on the right in the table below.



θ in degrees	θ in radians
$l = 2\pi r\left(\frac{\theta}{360}\right)$	$l = r\theta$
$A = \pi r^2\left(\frac{\theta}{360}\right)$	$A = \frac{1}{2}r^2\theta$

See Tables book pg 9

- **Example 1:** Find the area and length of a sector of radius 5cm with angle $\frac{2\pi}{3}$ radians at the centre.

  Solution:

  - We could just convert the angle to degrees and use the formulae from J.C. but let's see how to do it using the radians…..

  $l = r\theta$

  $l = (5)(\frac{2\pi}{3})$

  = 10.47cm

  Fill in 3.14 or use π on calculator.

  $A = \frac{1}{2}r^2\theta$

  $A = \frac{1}{2}(5)^2(\frac{2\pi}{3})$

  = 16.17cm^2

- **Example 2:** A roundabout is turning at a speed of 4 radians per second. A girl is standing on the outer edge, 1.5m from the centre. What is the speed of the girl in m/s?

  Solution:

  - In 1 second, the girl will have travelled a distance of:

  $$l = r\theta = 1.5(4) = 6m$$

  - That means her speed will be 6m/s.

  Note on Angular Speed:

  - The 4 rads/s in example 2 above is known as the angular speed.
  - We use the Greek letter ω to denote it.
  - We can now form a useful link between angular speed and linear speed.
  - If an object is moving on a circular path of radius r, with an angular speed of ω rad/s:
    - In one second, the angle subtended is ω radians.
    - If the object is travelling at a linear speed of v, then in one second, the particle travels a distance of v(1) = v
    - As $l = r\theta$, it follows that $v = r\omega$ or:

  $$v = \omega r$$

  ω must be in rad/s

- **Example 3:** An old LP player is spinning at 10 full turns a minute. A speck of dirt is 15cm from the centre. What speed is the dirt travelling in cm/s?

  Solution:

  - It completes 10 turns in a minute => it completes $\frac{1}{6}$ of a turn in 1 second.
  - $\frac{1}{6}$ of a turn is $\frac{1}{6}$ of 2π radians = $\frac{1}{6}$ of 2(3.14) radians = 1.05rads => ω = 1.05rads/s
  - Now, $v = \omega r$ => v = (1.05)(15) = 15.75cm/s.

Classwork Questions: Pg 125 Ex 7A Qs 1(ii)(iv)/2(iv)/4/6 - 10

Topic 30: Motion in a Horizontal Circle

- The moon orbits the earth at a steady speed, and the force of gravity keeps it on its orbital path.
- Although the moon is travelling at a steady speed, its velocity is constantly changing in direction.
- If the velocity is changing, then the moon must be undergoing an acceleration.
- This acceleration is known as centripetal acceleration.
- It can be shown that the direction of the acceleration is always in the direction of the centre of the circle.
- We will come back to the proof of these results at a later stage but for now, we will just use the results:

$$Acc = \omega^2 r \quad \text{or} \quad Acc = \frac{v^2}{r}$$

See Tables book pg 51

- As we know from a previous chapter F = ma, so it follows that the Force experienced by the object will be given by:

$$F = m\omega^2 r \quad \text{or} \quad F = \frac{mv^2}{r}$$

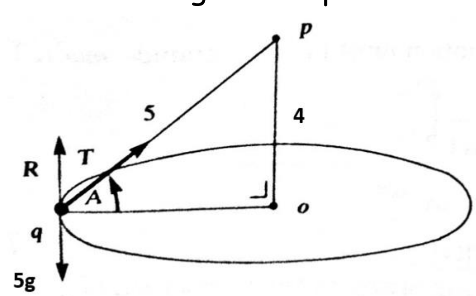
- **Example 1:** Pg 129 Ex 7B Q3

A light inextensible string of length 5 m is attached at one end to a fixed point P, which is 4 m above a smooth horizontal table. The other end is attached to a particle of mass 5 kg, which moves in a circle on the table. If the speed of the particle is 3 m/s, find:

i) The radius of the circle of motion
ii) The tension in the string
iii) The reaction at the table

Solution:

i) The diagram showing the forces acting on the particle is shown below:



- First, we will find |OQ|:

|PQ|^2 = |OP|^2 + |OQ|^2

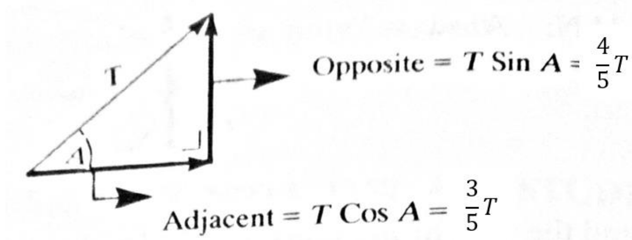
=> $(5)^2 = (4)^2$ + |OQ|^2

=> |OQ| = 3 m

=> $\sin A = \frac{OPP}{HYP} = \frac{4}{5}$ and $\cos A = \frac{ADJ}{HYP} = \frac{3}{5}$

ii)

- We now resolve T into horizontal and vertical components:



- As there is no vertical acceleration, so the forces up must be balancing the forces down:

=> $R + \frac{4}{5}T = mg$

=> $R + \frac{4}{5}T = 5(9.8)$

=> $R + \frac{4}{5}T = 49$

=> $5R + 4T = 245$……..Eqn 1

- In this case the centripetal force must be the $\frac{3}{5}T$

=> $F = \frac{mv^2}{r}$

=> $\frac{3}{5}T = \frac{(5)(3)^2}{3}$

=> $\frac{3}{5}T = 15$

=> $3T = 75$

=> $T = 25$ N

iii)

- From equation 1, we can now find the reaction force R:

$5R + 4T = 245$

=> $5R + 4(25) = 245$

=> $5R + 100 = 245$

=> $5R = 245 - 100$

=> $5R = 145$

=> $R = 29$ N

Classwork Questions: Pg 129 Ex 7B Qs 1/4/5/6

- Example 2: Pg 129 Ex 7B Q7

A marble of mass 10 kg is describing a horizontal circle on the smooth inside surface of a sphere. The radius of the sphere is 1.3 m and the centre of the circle of motion is 0.5 m below the centre of the sphere.

i) Find the radius of the circle of motion.

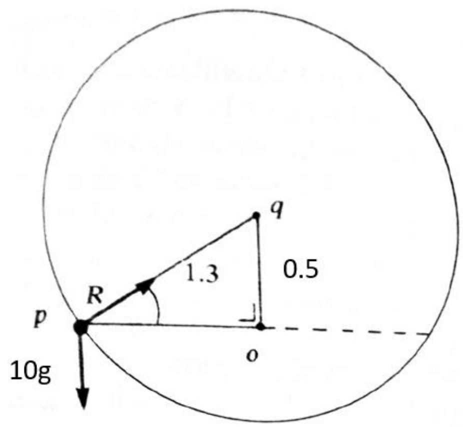
ii) Find the normal reaction between the marble and the sphere.

iii) Find the constant angular velocity ω of the marble.

Solution

i)

- The diagram showing forces:



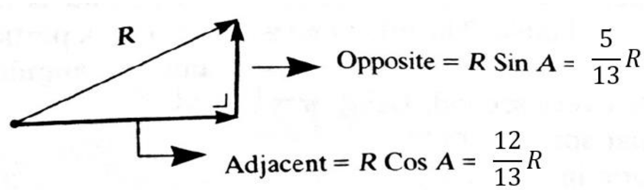
- Again, we find |OP| using Pythagoras:

|OP| = 1.2

=> => $\sin A = \frac{OPP}{HYP} = \frac{0.5}{1.3} = \frac{5}{13}$ and $\cos A = \frac{ADJ}{HYP} = \frac{1.2}{1.3} = \frac{12}{13}$

ii)

- The forces acting on the particle are its weight 10g and the reaction force R.
- In this case, the reaction force R is perpendicular to the surface of the sphere, which is along the line PQ.
- Resolving R vertically and horizontally:



- In this example again, there is no vertical acceleration, so the forces up must equal the forces down:

$$\frac{5}{13}R = 10g$$

=> $5R = 130g$

=> $R = \frac{130g}{5} = 254.8$ N

iii)

- As before, the force towards the centre of motion is given by $m\omega^2 r$:

$$\Rightarrow \frac{12}{13}R = m\omega^2 r = (10)\omega^2(1.2)$$

- And we can now fill in our expression for R:

$$\Rightarrow \frac{12}{13}(\,254.8\,) = 12\omega^2$$

$$\Rightarrow 235.2 = 12\omega^2$$

$$\Rightarrow \omega^2 = \frac{235.2}{12} = 19.6$$

$$\Rightarrow \omega = \sqrt{19.6} = 4.42 \text{ rads/s}$$

Classwork Questions: Pg 129 Ex 7B Qs 8 – 11

- We will now look at some more complicated examples.

- Example 3: Pg 132 Ex 7C Q2

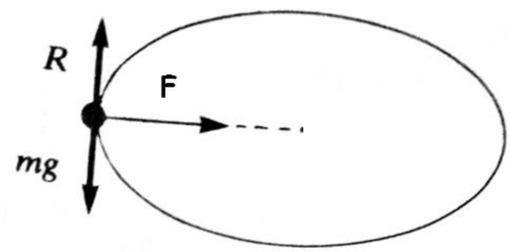
  i) A car turns around a bend on a horizontal road. The bend forms the arc of a circle of radius 24 m. If the coefficient of friction between the car's tyres and the road is $\frac{2}{3}$, find the maximum speed with which the car can safely take the bend.

  ii) If, in wet weather, the maximum speed is 7 m/s, find the coefficient of friction between the tyres and the wet road.

  Solution:

  i)

- We will let 'm' represent the mass of the car.
- The forces acting on the car are shown below:



- Note that the friction force 'F' will act in towards the centre of the circle as the car will tend to slide out.
- There is no vertical acceleration again in this situation so:

$$R = mg$$

$$\Rightarrow R = 9.8m$$

- The centripetal force in this case, is the of force of friction so:

$$\mu R = \frac{mv^2}{r}$$
$$\Rightarrow \frac{2}{3}(9.8m) = \frac{m(v)^2}{24}$$
$$\Rightarrow \frac{98}{15} = \frac{(v)^2}{24}$$
$$\Rightarrow 15v^2 = 98(24)$$
$$\Rightarrow v^2 = 156.8$$
$$\Rightarrow v = \sqrt{156.8} = 12.52 \text{ m/s}$$

ii) If v = 7 m/s, our equations will remain the same as above and we can solve for μ:

$$\Rightarrow \mu R = \frac{mv^2}{r}$$
$$\Rightarrow \mu(9.8m) = \frac{m(7)^2}{24}$$
$$\Rightarrow 9.8\mu = \frac{49}{24}$$
$$\Rightarrow \mu = \frac{5}{24}$$

Day 1: Classwork Questions: Pg 132 Ex 7C Qs 1/3/5/6

Day 2: Classwork Questions: Pg 132 Ex 7C Qs 8/9/10

Topic 31: Motion in a Vertical Circle

- When objects move on paths of vertical circles, we need to combine some results we used in a previous chapter i.e. the law of conservation of energy i.e.

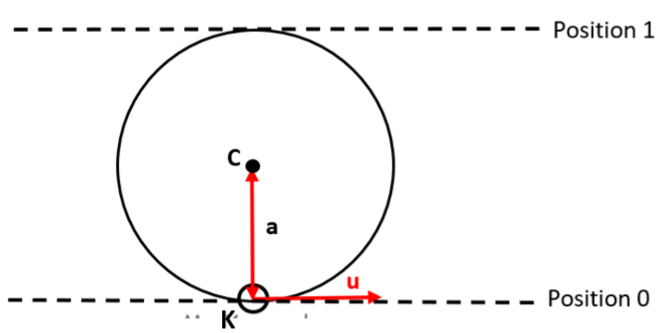
$$mgh + \frac{1}{2}mv^2 = constant$$

- **Example 1:**

A ring K of mass m is threaded on a smooth circular wire, centre C and of radius a, which is fixed in a vertical position. The ring is projected horizontally with initial speed u from the lowest point on the wire. Show that the ring will just reach the uppermost point on the wire if $u = \sqrt{4ga}$. Find the magnitude of the reaction at the wire when the line CK makes an angle $60°$ with the downward vertical from C, if $u = \sqrt{4ga}$.

Solution:

- As always, we will start by drawing a diagram of the situation to help visualise what's going on:



- We now use the Law of Conservation of Energy, with Position 0 and Position 1 defined as in the diagram above.
- The first part of the question says the particle "just reaches the uppermost point", which means it's final speed at that point (Position 1) will be 0.
- So:

$$mgh_0 + \frac{1}{2}m{v_0}^2 = mgh_1 + \frac{1}{2}m{v_1}^2$$

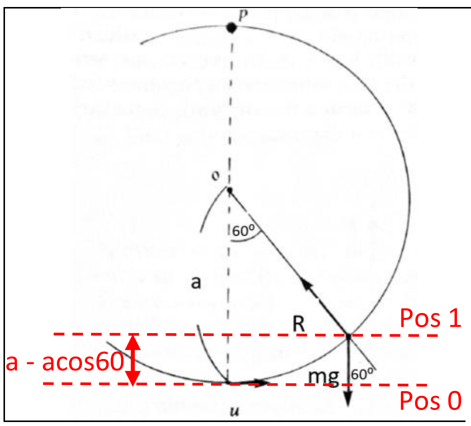
=> $mg(0) + \frac{1}{2}mu^2 = mg(2a) + \frac{1}{2}m(0)^2$

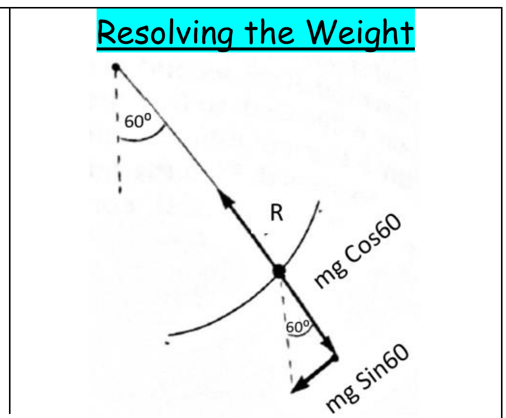
=> $mu^2 = 4mga$ (multiplying across by 2)

=> $u^2 = 4ga$ (dividing across by m and rearranging)

=> $u = \sqrt{4ga}$ Q.E.D.

- Let's look at the forces on the ring at an instant when it makes an angle of $60°$ with the downward vertical.





- Letting 'v' be the speed of the ring at position 1, then the resultant centripetal force must have magnitude $\frac{mv^2}{r}$
- This force is $R - mgcos60$, so:

=> $R - mgcos60 = \frac{mv^2}{a}$

=> $R - mg(\frac{1}{2}) = \frac{mv^2}{a}$..................Equation 1

- We now reapply the Law of Conservation of Energy between the two positions shown in the diagram above:

$$mgh_0 + \frac{1}{2}m{v_0}^2 = mgh_1 + \frac{1}{2}m{v_1}^2$$

=> $mg(0) + \frac{1}{2}m(\sqrt{4ga})^2 = mg(a - acos60) + \frac{1}{2}m(v)^2$

=> $\frac{1}{2}m(4ga) = mg(a - \frac{a}{2}) + \frac{1}{2}m(v)^2$

=> $2mga = mga - mg\frac{a}{2} + \frac{1}{2}m(v)^2$

=> $4ga = 2ga - ga + v^2$ (multiplying across by 2 and dividing by m)

=> $v^2 = 3ga$ ....Equation 2

- We now substitute Equation 2 into Equation 1 above:

=> Equation 1 becomes: $R - mg(\frac{1}{2}) = \frac{m(3ga)}{a}$

=> $R - mg\left(\frac{1}{2}\right) = 3mg$

=> $2R - mg = 6mg$ (multiplying across by 2)

=> $2R = 7mg$

=> $R = \frac{7mg}{2}$ N

Day 1: Classwork Questions: Pg 135 Ex 11D Qs 1/2/4 and then try Q5

Day 2: Classwork Questions: Pg 135 Ex 11D Qs 6/7/10/11

- Topic 32: Hooke's Law
  - You might recall learning about Hooke's Law on your Junior Cert Science course:

> The extension of a spring, or elastic material is directly proportional to the force causing the extension.

- When carrying out the experiment in the lab, you might remember having to start by measuring the length of the spring, without any force applied.
- This is known as the natural length of the spring, and is denoted l_0.
- Once the force was applied, the next step was to subtract the natural length from the length of the spring (l), to measure the extension.
- So, the extension would have been $l - l_0$.
- Finally, when you graphed all the forces versus the corresponding extensions, the result was a straight line graph going through the origin, which meant there was a directly proportional relationship between the two quantities.
- Mathematically, that means that the Force must be equal to some multiple of the extension, or: $F = k(l - l_0)$

Where k = the elastic constant with units N/m

- Example 1: An elastic string has a natural length of 2.5 m and elastic constant of 64 N/m. Find
  - i) the tension that produces an extension of 0.3 m
  - ii) the length of the string when a force of 96 N is applied.

Solution:

i) In this case, we are given the extension, so: $F = k(l - l_0)$ => $F = (64)(0.3)$ => $F = 19.2\ N$	ii) $F = k(l - l_0)$ => $96 = 64(l - 2.5)$ => $96 = 64l - 160$ => $64l = 256$ => $l = 4$ m.

- Example 2: Pg 138 Ex 7E Q3

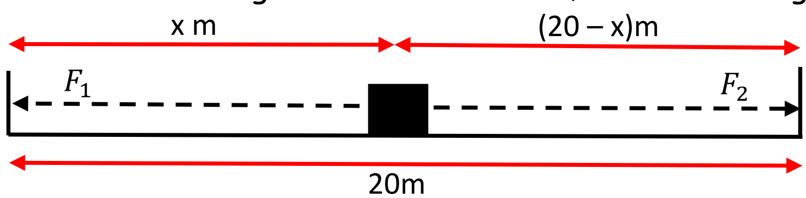
  A particle rests on a smooth horizontal table. It is attached to the ends of two horizontal elastic strings, the other ends being fixed to two vertical walls. The left string has natural length 1 m and elastic constant 2 N/m; the other has natural length 1 m and elastic constant 4 N/m. The walls are 20 m apart. Find:

  i) the resultant force on the particle when it is midway between the walls.

  ii) the position where the particle would be in equilibrium.

Solution:

- We will start with a diagram of the situation, to see what's going on:



i) If the particle is midway between the walls, then x = 10 and the resultant force on the particle will be $F_1 - F_2$, so we'll start by calculating F_1 and F_2:

$F_1 = k(l - l_0)$ $\quad\quad$ $F_2 = k(l - l_0)$

=> $F_1 = 2(10 - 1)$ $\quad\quad$ => $F_2 = 4(10 - 1)$

=> $F_1 = 18$ $\quad\quad$ => $F_2 = 36$

=> $F_1 - F_2 = 18 - 36 = -18$

=> Resultant force on the particle = 18 N

ii) If the particle is in equilibrium, then $F_1 = F_2$.

- We will let 'x' be the distance from the particle to the left wall, so 20 – x will be the distance to the right hand wall.
- We can now use Hooke's Law on both sides of the particle:

$F_1 = k(l - l_0)$ $\quad\quad$ $F_2 = k(l - l_0)$

=> $F_1 = 2(x - 1)$ $\quad\quad$ => $F_2 = 4(20 - x - 1)$

=> $F_1 = 2x - 2$ $\quad\quad$ => $F_2 = 76 - 4x$

=> $2x - 2 = 76 - 4x$

=> $2x + 4x = 76 + 2$

=> $6x = 78$

=> $x = 13$ m

=> Particle is 13 m from the left wall, or 7 m from the right wall.

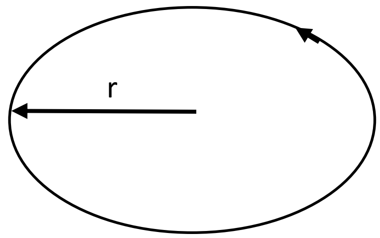
Classwork Questions: Pg 138/139 Ex 7E Qs 1(ii)(iii)/2(ii)(iii)/4/5

- Example 3: Pg 139 Ex 7E Q6

  A particle of mass 1 kg is attached to one end of an elastic string of natural length 1 m and elastic constant 50 N/m. The other end of the string is fixed to a point on a smooth horizontal table. The particle moves in a circle of radius 2 m. Find its angular speed.

  Solution:

  - By letting r be the radius of the circle, a diagram for this particular problem is:



  - If the particle is moving in a circle, then the string will be extended out to a length of 2 due to the centripetal force acting on it.
  - Using Hooke's Law to get one expression for this force gives:

$F = k(l - l_0)$

=> $F = 50(2 - 1)$

=> $F = 50$

  - We learned in the last chapter that this centripetal force is $m\omega^2 r$, so we can equate the two expressions now to find ω:

$50 = m\omega^2 r$

=> $50 = (1)\omega^2(2)$

=> $2\omega^2 = 50$

=> $\omega^2 = 25$

=> $\omega = \sqrt{25} =$ 5 rad/s

Classwork Questions: Pg 139/140 Ex 7E Qs 8 – 11

Revision Questions and Test