Topic 8: Calculus (Differentiation)

1) The Basics:

a) Differentiating Expressions:

Notes:

- Symbols: $\frac{dy}{dx}$ or $f'(x)$
- The derivative of a constant = 0
- In general, to differentiate, we:

Multiply by the power and reduce the power by 1.

Examples: Differentiate i) y = $3x^2$ - 5x + 2 and ii) y = $\sqrt{x}$, @ the point (1, 3).

i) If y = $3x^2$ - 5x + 2

=> $\frac{dy}{dx} = 2(3)x^{2-1} - 5x^{1-1} + 0$

=> $\frac{dy}{dx} = 6x^1 - 5x^0 + 0$

=> $\frac{dy}{dx} = 6x^1 - 5(1)$

=> $\frac{dy}{dx} = 6x - 5$

ii) If y = $\sqrt{x}$, rewrite y first:

y = $x^{\frac{1}{2}}$

=> $\frac{dy}{dx} = \frac{1}{2}x^{\frac{1}{2}-1}$ (rule 1)

=> $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$

@ the point (1, 3) means x = 1

=> $\frac{dy}{dx} = \frac{1}{2}(1)^{-\frac{1}{2}} = \frac{1}{2}$

b) Second Derivative: (Differentiating twice)

Examples:

i) $y = 3x^3 - 4x^2 + 5x$

=> $\frac{dy}{dx} = 9x^2 - 8x + 5$

=> $\frac{d^2y}{dx^2} = 18x - 8$

ii) $f(x) = 2x^4 + 3x^2 - 3x + 2$

=> $f'(x) = 8x^3 + 6x - 3$

=> $f''(x) = 24x^2 + 6$

c) Slopes of Tangents:

Slope of the Tangent to the Curve = $\frac{dy}{dx}$

Example 1:

If $y = 3x^2 + 5x - 8$, find the slope of the tangent to the curve at the point (-1, 2).

$y = 3x^2 + 5x - 8$

=> $\frac{dy}{dx} = 6x + 5$

@ (-1, 2)

=> $\frac{dy}{dx} = 6(-1) + 5 = -1$

Example 2: If $y = 2x^3 - 4x + 1$, find the equation of the tangent to the curve at the point (-2, -6).

Find slope first: $\frac{dy}{dx} = 6x^2 - 4$

Slope @ (-2, -6): $\frac{dy}{dx} = 6(-2)^2 - 4 = 20$

Find the equation using equation formula and the point (-2,-6):

$$y - y_1 = m(x - x_1)$$
$$y - (-6) = 20(x - (-2))$$
$$y + 6 = 20(x + 2)$$
$$y = 20x + 34$$

2) Product Rule/Quotient Rule:

a) Product Rule:

Notes:

- Used when two functions are **multiplied** together

If $y = uv$ then $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

See Tables pg25

Example: If y = (x^2 + 3)(2x - 1), find $\frac{dy}{dx}$.

Let u = x^2 + 3 and v = 2x - 1

$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

=> $\frac{dy}{dx} = (2x - 1)(2x) + (x^2 + 3)(2)$

=> $\frac{dy}{dx} = 4x^2 - 2x + 2x^2 + 6$

=> $\frac{dy}{dx} = 6x^2 - 2x + 6$

b) Quotient Rule:

Notes:

- Used when two functions are **divided** by each other

If $y = \frac{u}{v}$ then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

See Tables pg25

Example: If y = $\frac{3x - 4}{x^2 + 1}$, find $\frac{dy}{dx}$.

Let u = 3x - 4 and v = x^2 + 1.

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

=> $\frac{dy}{dx} = \frac{(x^2 + 1)(3) - (3x - 4)(2x)}{(x^2 + 1)^2}$

=> $\frac{dy}{dx} = \frac{3x^2 + 3 - 6x^2 + 8x}{(x^2 + 1)^2}$

=> $\frac{dy}{dx} = \frac{-3x^2 + 8x + 3}{(x^2 + 1)^2}$

3) Chain Rule:

a) Chain Rule:

Notes:

- Used for functions embedded in each other.
- Start on the outside and then work on the inside.

Examples: Differentiate the following i) f(x) = $(3x^2 - 2x)^3$ ii) g(x) = $\sqrt{2x^2 - 3}$

i) Differentiating the power 1st:

= $3(3x^2 - 2x)^{3-1}$

= $3(3x^2 - 2x)^2$

- Now differentiate the function inside i.e. $3x^2 - 2x$:

= $2(3x^{2-1}) - 2(1)x^{1-1}$

= $6x - 2x^0$

= $6x - 2$

- Now multiply both derivatives together:

= $(6x - 2)(3)(3x^2 - 2x)^2$

Finally, we tidy up the terms at the front to get:

= $(18x - 6)(3x^2 - 2x)^2$

ii) Rewrite the function:

g(x) = $(2x^2 - 3)^{\frac{1}{2}}$

- Differentiate the power on the outside:

= $\frac{1}{2}(2x^2 - 3)^{\frac{1}{2} - 1}$

= $\frac{1}{2}(2x^2 - 3)^{-\frac{1}{2}}$

- Differentiate the function inside i.e. $2x^2 - 3$:

= $2(2)x^{2-1} - 0$

= $4x$

Now, multiply the two derivatives together:

= $4x(\frac{1}{2}(2x^2 - 3)^{-\frac{1}{2}})$

- Tidying up gives:

= $2x(2x^2 - 3)^{-\frac{1}{2}}$

- Or we could rewrite as:

= $\frac{2x}{\sqrt{(2x^2 - 3)}}$

b) Trigonometric Functions:

Notes:

- To differentiate we use the Tables pg 25:

$f(x)$	$f'(x)$
$\cos x$	$-\sin x$
$\sin x$	$\cos x$
$\tan x$	$\sec^2 x$

- Use the Chain Rule then following the order below:

Remember PTA.
P = Power, T = Trig, A = Angle

Examples: Differentiate i) cos2x and ii) $\tan^3(x^2 + 3)$.

i) y = cos2x

=> $\frac{dy}{dx} = (-\sin 2x)(2)$

=> $\frac{dy}{dx} = -2\sin 2x$

ii) y = $\tan^3(x^2 + 3)$

=> $\frac{dy}{dx} = (3\tan^2(x^2 + 3))(\sec^2(x^2 + 3))(2x)$

=> $\frac{dy}{dx} = 6x\tan^2(x^2 + 3)\sec^2(x^2 + 3)$

c) Log/Exponential Functions:

Notes:

- Use the Chain Rule.
- To differentiate we use the Tables pg 25:

$f(x)$	$f'(x)$
$\ln x$	$\frac{1}{x}$
e^x	e^x
e^{ax}	ae^{ax}
a^x	$a^x \ln a$

Examples: Differentiate the following i) $f(x) = e^{-3x}$ ii) $f(x) = \log_e 4x$.

i) $f(x) = e^{-3x}$

We use the Chain Rule to differentiate the exponential function first, and then the power:

=> $f'(x) = e^{-3x}(-3)$

=> $f'(x) = -3e^{-3x}$

ii) $f(x) = \log_e 4x$

We use the Chain Rule to differentiate the log function first, and then the 4x:

=> $f'(x) = \frac{1}{4x} \times (4)$

=> $f'(x) = \frac{4}{4x}$

=> $f'(x) = \frac{1}{x}$

d) Inverse Trig Functions:

Notes:

- Use the Chain Rule.
- To differentiate we use the Tables pg 25:

$f(x)$	$f'(x)$
$\cos^{-1}\frac{x}{a}$	$-\frac{1}{\sqrt{a^2 - x^2}}$
$\sin^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{a^2 - x^2}}$
$\tan^{-1}\frac{x}{a}$	$\frac{a}{a^2 - x^2}$

Examples: Differentiate i) $\cos^{-1} 4x$ and ii) $\frac{\sin^{-1}\frac{x}{3}}{3x}$

i) Rewrite the function:

$y = \cos^{-1}\frac{4x}{1}$

Use the Chain Rule to differentiate the inverse Trig function first, and then the angle:

=> $\frac{dy}{dx} = -\frac{1}{\sqrt{(1)^2 - (4x)^2}} \times (4)$

=> $\frac{dy}{dx} = \frac{-4}{\sqrt{1 - 16x^2}}$

ii) Use Product Rule:

Let u = 3x and v = $\sin^{-1}\frac{x}{3}$

=> $\frac{du}{dx} = 3$ and $\frac{dv}{dx} = \frac{1}{\sqrt{(3)^2 - (x)^2}}$

=> $\frac{du}{dx} = 3$ and $\frac{dv}{dx} = \frac{1}{\sqrt{9 - x^2}}$

=> $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

=> $\frac{dy}{dx} = (\sin^{-1}\frac{x}{3})(3) + (3x)(\frac{1}{\sqrt{9 - x^2}})$

=> $\frac{dy}{dx} = 3\sin^{-1}\frac{x}{3} + \frac{3x}{\sqrt{9 - x^2}}$

4) Curve Sketching:

a) Increasing/Decreasing Functions:

Increasing => $\frac{dy}{dx} > 0$

Decreasing => $\frac{dy}{dx} < 0$

Example: Find the range of values of x for which the curve $f(x) = x^2 - 3x + 4$ is increasing.

$f'(x) = 2x - 3$

Increasing => $f'(x) > 0$

$2x - 3 > 0$

$2x > 3$

=> $x > \frac{3}{2}$

b) Max/Min Points (Turning Points):

Max/Min Points => $\frac{dy}{dx} = 0$

If $\frac{d^2y}{dx^2} < 0$, then it's a local maximum point.

If $\frac{d^2y}{dx^2} > 0$, then it's a local minimum point.

Example: Find the max/min points of the curve $f(x) = x^3 - 2x^2 + 4$.

$f'(x) = 3x^2 - 4x$

Max/Min => $f'(x) = 0$

$3x^2 - 4x = 0$

$x(3x - 4) = 0$

$x = 0 \quad or\ x = \frac{4}{3}$

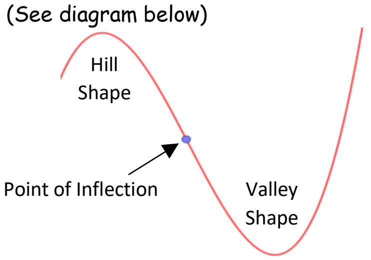
Sub into function at start to find y values of 4 and $\frac{76}{27}$.

=> Turning Points are $(0,4)$ and $(\frac{4}{3}, \frac{76}{27})$

c) Points of Inflection:

Notes:

- The point where a graph begins to change direction i.e. bend in a different direction, is known as a **point of inflection**. (See diagram below)



$$\frac{d^2y}{dx^2} = 0$$

Example: Find the point of inflection of the curve y = $x^3 - 3x^2 + 4$.

$\frac{d^2y}{dx^2} = 6x - 6$

=> $6x - 6 = 0$

=> $x = 1$

- So, the y value of the point of inflection will be:

y = $(1)^3 - 3(1)^2 + 4$

=> y = 2 => the point of inflection is (1, 2).

5) Implicit Differentiation:

Notes:

- Differentiating more than one variable with respect to another
- When differentiating y with respect to x, differentiate as normal and multiply by $\frac{dy}{dx}$.

Example: Find $\frac{dy}{dx}$ for the following curve: $x^2 + y^2 = 36$.

- The derivative of y^2 term is 2y but we have to multiply by $\frac{dy}{dx}$ as we are differentiating with respect to x:

=> $\frac{d(y^2)}{dx} = 2y.\frac{dy}{dx}$

- We now differentiate each term, with respect to x.

=> if $x^2 + y^2 = 36$

=> $2x + 2y.\frac{dy}{dx} = 0$

- We now rearrange to get $\frac{dy}{dx}$ on its own:

=> $2y.\frac{dy}{dx} = -2x$

=> $\frac{dy}{dx} = \frac{-2x}{2y}$

=> $\frac{dy}{dx} = \frac{-x}{y}$ (divide above and below by 2)

6) Rates of Change:

a) Max/Min Problems:

Steps:

1. Get an expression for quantity to be maximised/minimised.
2. Differentiate.
3. Let derivative = 0 and solve to find max/min value.
4. Sub max/min value back into expression from step 1, if needed.

Example: A farmer want to enclose a field with 100m of fencing. Find the maximum area of the field.

Let Width = x => Length = 50 – x

=> Area = L x W = x(50 - x) = $50x - x^2$

=> A = $50x - x^2$

=> $\frac{dA}{dx} = 50 - 2x$ (differentiating expression for area)

50 – 2x = 0

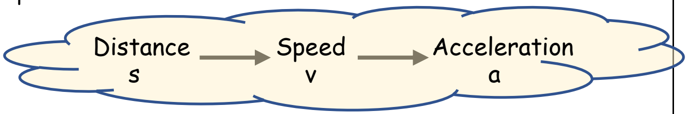
=> x = 25

=> Max Area will be $50(25) - (25)^2 = 625\text{m}^2$

b) Distance/Speed/Acceleration:

Tip:

Differentiate the expression for distance to get expressions for speed and acceleration.



Example: A body moves a distance given by the function $s = t^3 - 3t^2 + 7$, find the body's acceleration after 3 seconds.

Distance = $t^3 - 3t^2 + 7$

=> Speed = $3t^2 - 6t$ (differentiating distance expression)

=> Acceleration = $6t - 6$ (differentiating speed expression)

So, after 2 secs: Acceleration = $6t - 6$ = 6(3) – 6 = 12m/s^2.

c) Rates of Change:

Steps:

1. Write down what you're given.
2. Write down what you're trying to find.
3. Use a formula to link steps 1 and 2.
4. Use Implicit Differentiation to differentiate expression from step 3
5. Fill in what you were given and solve

Example 1: The rate of change of a radius of a circle is 2cm/s. Find the rate of change of the area of the circle when the radius is 3cm.

- First, we will write down the information we were given i.e. the rate of change of a radius

$$\frac{dr}{dt} = 2$$

- Now we write down an expression for what we are looking for i.e. the rate of change of area:

$$\frac{dA}{dt}$$

- And then we need a way to connect the two previous expressions together i.e. the formula for the area of a circle:

Area of a circle = $A = \pi r^2$

- We're looking for $\frac{dA}{dt}$ so we need to differentiate our expression for the area of a circle:

$A = \pi r^2$

=> $\frac{dA}{dt} = \pi 2r(\frac{dr}{dt})$ (using implicit differentiation)

=> $\frac{dA}{dt} = \pi 2r(2)$

=> $\frac{dA}{dt} = 4\pi r$

- Finally, we know the radius is 3cm

=> $\frac{dA}{dt} = 4\pi(3)$

=> $\frac{dA}{dt} = 12\pi$ cm^2/s

Example 2: The rate of change of volume of a sphere is 6 cm^3/s. Find the rate of change of the radius at r = 3.

- Again, we will write down what we've been given first:

$$\frac{dV}{dt} = 6$$

- We're looking for the rate of change of the radius:

$$\frac{dr}{dt}$$

- Now we need a formula to connect the two i.e. the volume of a sphere:

$$V = \frac{4}{3}\pi r^3$$

- And we differentiate to generate our expressions above:

$V = \frac{4}{3}\pi r^3$

=> $\frac{dV}{dt} = \frac{4}{3}\pi 3r^2 \frac{dr}{dt}$

=> $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

=> $6 = 4\pi r^2 \frac{dr}{dt}$

=> $\frac{dr}{dt} = \frac{6}{4\pi r^2}$

- Finally, we can fill in the value of r:

=> $\frac{dr}{dt} = \frac{6}{4\pi(3)^2}$

=> $\frac{dr}{dt} = \frac{6}{4\pi(9)}$

=> $\frac{dr}{dt} = \frac{6}{36\pi}$

=> $\frac{dr}{dt} = \frac{1}{6\pi}$ cm/s