Topic 6: Algebra

1) The Basics:

a) Adding / Subtracting Algebraic Expressions:

Notes:

- We can only add / subtract 'like terms'.
- 'Like terms' are terms that have the same letter part or the same variables
  e.g. 5d and -2d are 'like terms' but 5d and $5d^2$ are NOT 'like terms'

Example 1:

$4a + 5 + 2a - 3$

$= 6a + 2$

Example 2:

$3x^2y - 4y^2 - x^2y - 3y + 2y^2$

$= 2x^2y - 2y^2 - 3y$

b) Multiplying Expressions:

Notes:

- When multiplying we follow the order Signs, Numbers, Letters
- When multiplying the letters together we must remember the first law of indices.....$a^m \times a^n = a^{m+n}$ i.e. Add the Powers

Example 1: Multiplying Terms

$4a^2 \times 2a^5$ (Multiply signs...(+).(+) = +)

$=8a^7$ (Multiply Numbers & Add Powers)

Example 2: Removing Brackets

$2(g + 4)$

$= 2g + 8$

Example 3: Removing Brackets

$(2x - 3)(x + 2)$ ("Split and Repeat")

$= 2x(x + 2) - 3(x + 2)$

$= 2x^2 + 4x - 3x - 6$

$= 2x^2 + x - 6$

2) Adding/Subtracting Algebraic Fractions:

a) Adding Fractions:

Note:

- When adding/subtracting fractions together we find the common denominator and bring both terms up to the same denominator first.

Example 1:

$$\frac{x+3}{5} - \frac{2x-1}{3}$$

$$= \frac{3(x+3)}{15} - \frac{5(2x-1)}{15}$$

$$= \frac{3x+9}{15} - \frac{10x-5}{15}$$

$$= \frac{3x+9-(10x-5)}{15}$$

$$= \frac{3x+9-10x+5}{15}$$

$$= \frac{-7x+14}{15}$$

b) Subtracting Fractions:

Example 2:

$$\frac{5}{x+3} - \frac{2}{x-4}$$

$$= \frac{5(x-4)}{(x+3)(x-4)} - \frac{2(x+3)}{(x+3)(x-4)}$$

$$= \frac{5(x-4)-2(x+3)}{(x+3)(x-4)}$$

$$= \frac{5x-20-2x-6}{(x+3)(x-4)}$$

$$= \frac{3x-26}{(x+3)(x-4)}$$

Note: A shortcut we can use when doing the type of questions above:

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}$$

3) Pascal's Triangle/Binomial Expansion:

a) Pascal's Triangle:

Notes:

$(x + y)^1 = 1x + 1y$

$(x + y)^2 = 1x^2 + 2xy + 1y^2$

$(x + y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$

$(x + y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$

Example: Use Pascal's Triangle to expand $(3a + 2)^4$.

- We have a power of 4 here so we'll be following the 4th line of Pascal's Triangle above i.e. $(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$
- In this case, our x = 3a and y = 2 so we can fill those in now......

$(3a + 2)^4 = 1(3a)^4 + 4(3a)^3(2) + 6(3a)^2(2)^2 + 4(3a)(2)^3 + 1(2)^4$

$= 81a^4 + 216a^3 + 216a^2 + 96a + 16$

b) Binomial Expansion:

- The expansion of $(x + y)^n$ is given by:

$${}^nc_0 x^n y^0 + {}^nc_1 x^{n-1} y^1 + \cdots + {}^nc_n x^0 y^n$$

Example: Expand $(1 + 5y)^4$.

$(1 + 5y)^4 = {}^4c_0 (1)^4 (5y)^0 + {}^4c_1 (1)^3 (5y)^1 + \cdots + {}^4c_4 (1)^0 (5y)^4$

$= 1 + 4(5y) + 6(1)(25y^2) + 4(1)(125y^3) + 1(1)(625y^4)$

$= 1 + 20y + 150y^2 + 600y^3 + 625y^4$

4) Multiplication of Algebraic Fractions:

Example 1: Simplify $\frac{x-2}{x} \times \frac{3x^2}{x^2-4}$.

$$\frac{x-2}{x} \times \frac{3x^2}{x^2-4}$$

$$= \frac{x-2}{x} \times \frac{3x^2}{(x-2)(x+2)} \quad \text{(Factorising the } x^2 - 4\text{)}$$

$$= \frac{1}{1} \times \frac{3x}{x+2} \quad \text{(Cancelling the x - 2 and the x on diagonals)}$$

$$= \frac{3x}{x+2}$$

Example 2: Simplify $\frac{5x^2-45}{3x^2-7x+4} \times \frac{8x(3x-4)}{2x^2-6x}$.

$$\frac{5x^2-45}{3x^2-7x+4} \times \frac{8x(3x-4)}{2x^2-6x}$$

$$= \frac{5(x^2-9)}{(3x-4)(x-1)} \times \frac{8x(3x-4)}{2x(x-3)}$$

$$= \frac{5(x-3)(x+3)}{(3x-4)(x-1)} \times \frac{8x(3x-4)}{2x(x-3)}$$

$$= \frac{5(x+3)}{(x-1)} \times \frac{4}{1}$$

$$= \frac{20x+60}{x-1}$$

5) Division of Algebraic Expressions/Fractions:

a) Dividing Expressions:

Tip: Can we factorise the numerator or the denominator?

Example:

$$\frac{2x+6}{x^2-9} = \frac{2(x+3)}{(x+3)(x-3)} = \frac{2}{x-3}$$

b) Long Division:

Note: Remember: "Daddy, Mammy, Sister Brother", which stands for Divide, Multiply, Subtract, Bring Down

Example:

Simplify $\frac{x^3-5x^2+10x-12}{x-3}$.

```
              x² - 2x + 4
       ____________________________
x - 3 ) x³ - 5x² + 10x - 12
      - x³ ± 3x²
             -2x² + 10x
             ±2x² ∓ 6x
                     4x - 12
                   - 4x ± 12
                          0
```

e) Unknown Coefficients:

Example: If $x^2 + px + r$ is a factor of $x^3 + 2px^2 + 9x + 2r$, find p,q,r.

```
                   x  +  p
               ______________________________
x² + px + r  )  x³ + 2px² + 9x + 2r
          (-)   x³ + px²  + rx
                      px² + (9 - r)x + 2r
                 (-)  px² + p²x      + pr
                      (9 - r - p²)x + (2r - pr)
```

- $x^2 + px + r$ is a factor, so we should have no remainder.
- If we let each part of the remainder above = 0, then we can use the two equations to solve for p and r:

$2r - pr = 0$	and $9 - r - p^2 = 0$
=> $2r = pr$	$9 - r - (2)^2 = 0$
=> $p = 2$	$9 - r - 4 = 0$
	=> $5 - r = 0$
	=> $r = 5$

c) Dividing Algebraic Fractions 1:

Example: Simplify $\frac{3x-2}{x^2+5x+6} \div \frac{3x^2-2x}{x(x+2)}$.

$$\frac{3x-2}{x^2+5x+6} \div \frac{3x^2-2x}{x(x+2)}$$

$$= \frac{3x-2}{x^2+5x+6} \times \frac{x(x+2)}{3x^2-2x}$$

$$= \frac{3x-2}{(x+2)(x+3)} \times \frac{x(x+2)}{x(3x-2)}$$

$$= \frac{1}{x+3}$$

d) Dividing Algebraic Fractions 2:

Sometimes we can have a fraction in the numerator or the denominator.

Example 1: Simplify $\frac{1-\frac{4}{a^2}}{1+\frac{2}{a}}$.

- There are two different ways we can approach these types of questions:

Method 1: (Tidy up numerator and denominator to a single fraction first)

$$= \frac{\frac{a^2-4}{a^2}}{\frac{a+2}{a}}$$

$$= \frac{(a+2)(a-2)}{a^2} x \frac{a}{a+2}$$

$$= \frac{a-2}{a}$$

Method 2: (Multiply above and below by something)

$$= \frac{1-\frac{4}{a^2}}{1+\frac{2}{a}} \times \frac{a^2}{a^2}$$

$$= \frac{a^2-4}{a^2+2a}$$

$$= \frac{(a-2)(a+2)}{a(a+2)} = \frac{a-2}{a}$$

6) Factorising and Manipulation of Formulae:

a) Factorising:

1. Taking out the HCF (taking out what's common)

e.g.s i) $2x - 10 = 2(x - 5)$ ii) $3x^2 - 18x = 3x(x - 6)$

2. Grouping (always has 4 terms)

e.g.s i) $ax + ay + bx + by$
= $a(x + y) + b(x + y)$
= $(x + y)(a + b)$

ii) $3p - 3q - pk + kq$
= $3(p - q) - k(p - q)$
= $(p - q)(3 - k)$

3. Quadratic (always has 3 terms x^2, x, a)

e.g.s i) $x^2 + 5x + 6$
= $(x + 3)(x + 2)$

ii) $x^2 - 3x - 18$
= $(x - 6)(x + 3)$

4. Difference of 2 Squares (always 2 terms with a '-' between)

Note: Watch for square numbers: 1, 4, 9, 16, 25, 36, 49, 64, 81….

e.g.s i) $x^2 - 9y^2$
= $(x)^2 - (3y)^2$
= $(x - 3y)(x + 3y)$

ii) $16a^2 - 25b^2$
= $(4a)^2 - (5b)^2$
= $(4a - 5b)(4a + 5b)$

5. Sum/Difference of 2 Cubes

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$
$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

Examples: Factorise: i) $64p^3 - 27r^3$ and ii) $5x^3 - 625y^3$

i) $64p^3 - 27r^3$
= $(4p)^3 - (3r)^3$
= $(4p - 3r)[(4p)^2 + (4p)(3r) + (3r)^2]$
= $(4p - 3r)(16p^2 + 12pr + 9r^2)$

ii) $5x^3 - 625y^3$
= $5(x^3 - 125y^3)$
= $5[(x)^3 - (5y)^3]$
= $5(x - 5y)(x^2 + 5xy + 25y^2)$

b) Manipulation of Formulae:

Steps:

1) Get rid of any brackets, fractions or square roots.
2) Bring all terms with the letter you want to the LHS and move everything else to the RHS.
3) Factorise out the letter you want (if necessary).
4) Divide both sides to leave the letter you want on the LHS.

Example: Write r, in terms of p and q.

$$\sqrt{\frac{p}{r-q}} = p$$

$\Rightarrow \left(\sqrt{\frac{p}{r-q}}\right)^2 = (p)^2$ (Squaring both sides to get rid of $\sqrt{\ }$)

$\Rightarrow \frac{p}{r-q} = p^2$

$\Rightarrow p = p^2(r - q)$ (Multiplying both sides by (r - q))

$\Rightarrow p = p^2r - p^2q$

$\Rightarrow -p^2r = -p - p^2q$ (Bringing term with r to LHS)

$\Rightarrow p^2r = p + p^2q$ (Changing all the signs)

$\Rightarrow r = \frac{p + p^2q}{p^2}$ (dividing both sides by p^2)

7) Solving Equations:

a) Solving Linear Equations: (x only)

Steps:

1. Remove all brackets and any fractions
2. Bring all terms with an 'x' to one side and numbers to the other side
3. Tidy up both sides by putting together 'like terms'.
4. Solve the simple equation remaining.

Example: $2(x - 3) = 4(x + 1)$

$2x - 6 = 4x + 4$
$2x - 4x = 4 + 6$
$-2x = 10$
$x = \frac{10}{-2}$
=> $x = -5$

b) Solving Linear Equations With Fractions:

Tip:

"Kill" all fractions first by multiplying all terms by something that ALL denominators divide into.

Example: Solve $\frac{2x-3}{4} + \frac{x+6}{5} = \frac{3}{2}$

In this case 20 will kill the fractions, so multiply across by 20:

$$\cancel{20}\left(\frac{2x - 3}{\cancel{4}}\right) + \cancel{20}\left(\frac{x + 6}{\cancel{5}}\right) = \cancel{20}\left(\frac{3}{\cancel{2}}\right)$$

$5(2x - 3) + 4(x + 6) = 10(3)$
$10x - 15 + 4x + 24 = 30$
$10x + 4x = 30 + 15 - 24$
$14x = 21$
$\Rightarrow x = \frac{21}{14} = \frac{3}{2}$

c) Solving Quadratic Eqns by factorising: (Eqns with an x^2)

Steps:

1. Bring all terms to the left-hand side (LHS) and leave '0' on the RHS
2. Factorise the LHS (See section on Factorising in previous tab)
3. If LHS can't be factorised the 'Quadratic Formula' needs to be used (See Example 3 on the right)
4. Let each factor be = 0
5. Solve the two simple equations to find the two answers.

Example 1: $x^2 - 3x - 18 = 0$

$(x - 6)(x + 3) = 0$
$x - 6 = 0$ or $x + 3 = 0$
=> $x = 6$ or $x = -3$

Example 2: $4x^2 - 25 = 0$

$(2x - 5)(2x + 5) = 0$
$2x - 5 = 0$ or $2x + 5 = 0$
=> $2x = 5$ or $2x = -5$
=> $x = \frac{5}{2}$ or $x = \frac{-5}{2}$

d) Solving Quadratic Eqns using the "-b Formula":

Note: This method can be used for ALL quadratic equations.

If $ax^2 + bx + c = 0$ is a quadratic equation, then the roots of the equation are given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

See Tables pg 20

Example 3: Solve $x^2 - 2x - 5 = 0$.

In this case: a = 1, b = -2 and c = -5

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-5)}}{2(1)}$

$\Rightarrow x = \frac{2 \pm \sqrt{24}}{2}$

$\Rightarrow x = 3.45$ or $x = -1.45$

e) Quadratic Eqns with fractions:

Example: Solve $\frac{2}{x+1} - \frac{3}{x-2} = \frac{5}{2}$.

Method 1: (Multiply across by common denominator)
In this case the common denominator would be $2(x+1)(x-2)$:

$$2(x+1)(x-2)\frac{2}{x+1} - 2(x+1)(x-2)\frac{3}{x-2} = 2(x+1)(x-2)\frac{5}{2}$$

2(x - 2)(2) - 2(x + 1)(3) = 5(x + 1)(x - 2)

4x - 8 - 6x - 6 = 5x^2 - 5x - 10

- 5x^2 + 3x - 4 = 0

5x^2 - 3x + 4 = 0.......and solve this as before.

Method 2: (Tidy up both sides into single fractions and cross multiply) (See Section 2 - Example 2)

$$\frac{2}{x+1} - \frac{3}{x-2} = \frac{5}{2}$$

$$\frac{2(x-2) - 3(x+1)}{(x+1)(x-2)} = \frac{5}{2}$$

$$\frac{-x-7}{(x+1)(x-2)} = \frac{5}{2}$$

$$2(-x-7) = 5(x+1)(x-2)$$

$$-2x - 14 = 5(x^2 - x - 2)$$

$$-2x - 14 = 5x^2 - 5x - 10$$

$$5x^2 - 3x + 4 = 0 \quad \text{etc.}$$

f) Forming Quadratic Equation from the roots:

Method 1:

Steps:

1. Let x = both of the roots.
2. Create two factors that are = 0.
3. Multiply the two factors together using "split and repeat".

Example: Find the quadratic equation with roots -1 and 3.

$$x = -1 \quad or \quad x = 3$$
$$x + 1 = 0 \quad or \quad x - 3 = 0$$
$$(x+1)(x-3) = 0$$
$$x(x-3) + 1(x-3) = 0$$
$$x^2 - 3x + x - 3 = 0$$
$$x^2 - 2x - 3 = 0$$

Method 2: Use the formula

Need to know to use this method.

$$x^2 - (sum\ of\ roots)x + (product\ of\ roots) = 0$$

Example: Find the quadratic equation with roots -1 and 3.

$$x^2 - (sum\ of\ roots)x + (product\ of\ roots) = 0$$
$$x^2 - (-1+3)x + ((-1)(3)) = 0$$
$$x^2 - 2x - 3 = 0$$

8) Simultaneous Equations:

a) Two Linear Equations:

Steps:

1. Choose a variable to eliminate e.g. 'y'
2. Multiply one or both equations to make no. in front of y the same
3. Multiply the 2nd equation by -1, if necessary, to make signs in front of 'y' different.
4. Add the two equations to eliminate 'y' and solve for 'x'.
5. Put x back into one of the equations to find y.

Example: Solve the equations below:

$A\colon\ 2x - 3y = 7$

$B\colon\ 3x + 2y = 4$

$Ax2\colon\ 4x - 6y = 14$ (mult by 2 to get 6 in front of y)

$Bx3\colon\ 9x + 6y = 12$ (mult by 3 to get 6 in front of y)

$13x = 26$ (adding both equations together)

$\Rightarrow x = \frac{26}{13}$ (dividing both sides by 13)

$\Rightarrow x = 2$

Putting x into A:

$A\colon\ 2x - 3y = 7$

$\Rightarrow 2(2) - 3y = 7$

$\Rightarrow 4 - 3y = 7$

$\Rightarrow -3y = 7 - 4$

$\Rightarrow -3y = 3$

$\Rightarrow y = \frac{3}{-3}$ (dividing both sides by -3)

$\Rightarrow y = -1$

b) One Linear, One Quadratic:

Steps:

1. Use the linear equation to get one variable on its own.
2. Sub this into the quadratic equation.
3. Multiply out and solve the resulting quadratic equation.
4. Sub your two values back into the expression from step 1.

Example: Solve the equations x - y = 2 and 2x^2 + y^2 = 36.

L: x - y = 2

C: 2x^2 + y^2 = 36

Step 1: Use the linear equation to get one variable on its own:

L: x - y = 2

=> x = y + 2 *

Step 2: Substitute our expression for x into equation C:

C: 2x^2 + y^2 = 36

=> 2(y + 2)^2 + y^2 = 36

Step 3: Multiply out and solve the resulting quadratic equation:

=> 2(y^2 + 4y + 4) + y^2 = 36

=> 2y^2 + 8y + 8 + y^2 - 36 = 0

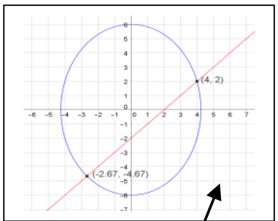
=> 3y^2 + 8y - 28 = 0

(3y + 14)(y - 2) = 0

=> 3y + 14 = 0 OR y - 2 = 0

3y = -14 OR y = 2

y = $\frac{-14}{3}$



Step 4: Sub your two values back into the expression from step 1:

x = y + 2 *

When y = 2

x = 2 + 2

x = 4

When y = $\frac{-14}{3}$

x = $\frac{-14}{3}$ + 2

x = $\frac{-8}{3}$

- So, our two solutions are: (4, 2) and ($\frac{-8}{3}$, $\frac{-14}{3}$).

c) Simultaneous Equations with 3 Unknowns: **Steps:** 1. Take equations in pairs and eliminate the same variable each time e.g. Solve A and B and eliminate z, and then Solve B and C and eliminate z 2. Solve two resulting equations for x and y values 3. Sub back into A, B or C to find z. **Example:** Solve A: $x + y + z = 9$ B: $2x + 3y + z = 16$ C: $3x - 4y + 2z = 1$ **Solving A and B to eliminate z:** A: $x + y + z = 9$ B x -1: $-2x - 3y - z = -16$ $-x - 2y = -7$………….Call this equation D **Solving B and C to eliminate z:** B x 2: $4x + 6y + 2z = 32$ C x -1: $-3x + 4y - 2z = -1$ $x + 10y = 31$…………..Call this equation E - We now solve equations D and E to find x and y **Solving D and E to eliminate z:** D: $-x - 2y = -7$ E: $x + 10y = 31$ $8y = 24$ $\Rightarrow y = 3$	**Now substitute y back into D or E:** D: $-x - 2y = -7$ $-x - 2(3) = -7$ $-x = -7 + 6$ $-x = -1$ $\Rightarrow x = 1$ **Finally, substitute x and y into A, B or C to find z:** A: $x + y + z = 9$ $(1) + (3) + z = 9$ $4 + z = 9$ $z = 9 - 4$ $\Rightarrow z = 5$ - So, the solution is (1, 3, 5).

9) Word Problems:

Tips:

1. Read the question a couple of times before attempting it.
2. Underline any Mathematical key words e.g. sum, product, total.
3. Let 'x' be what you are looking for, if there is one unknown. Use 'x' and 'y' for two unknowns.
4. Form an equation.
5. Solve the equation.
6. If you are unable to form an equation, try using "trial and improvement" to solve the problem. You need to show all trials and workings.
7. Check your answer(s).

Example 1: Find two consecutive natural numbers whose sum is 83. • Keywords: consecutive, natural and sum • Let x = 1st number, so that means $x + 1$ = 2nd number • Their sum is 83 ('sum' means they add to 83) $\Rightarrow x + x + 1 = 83$ (equation formed) $\Rightarrow 2x + 1 = 83$ $\Rightarrow 2x = 83 - 1$ $\Rightarrow 2x = 82$ $\Rightarrow x = 41$ (dividing both sides by 2) $\Rightarrow$ second number is $x + 1 = 42$ • Check…..41 + 42 = 83	**Example 2:** A shop sells 50 sofas in a week. A leather sofa costs €1000 and a fabric sofa costs €750. The shop sells €42,500 worth of sofas. How many of each type are sold? • Let x = no. of leather sofas and y = no. of fabric sofas • Total number of sofas = 50 $\Rightarrow x + y = 50$ (first equation formed) • Total money = €42,500 $\Rightarrow 1000x + 750y = 42500$ (second equation formed) • Can solve the 2 simultaneous equations now to find x and y. (See Section 5a)

10) Inequalities:

a) Solving Inequalities:

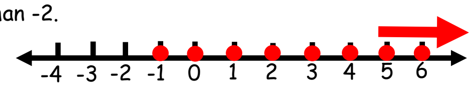
Notes:

- Need to know the types of numbers (See Arithmetic 1b)
- Same rules as solving linear equations (See Algebra 4a)
- One difference: if you have to multiply/divide both sides of an inequality by a **NEGATIVE** number, we must **CHANGE THE DIRECTION** of the inequality.

Example 1: Graph the solution to $3 - 4x < 11,\ x \in Z$.

$3 - 4x < 11$
$-4x < 11 - 3$
$-4x < 8$
$\frac{-4x}{-4} < \frac{8}{-4}$ (dividing both sides by -4)
$x > -2$ (Note sign change because divided by -4)

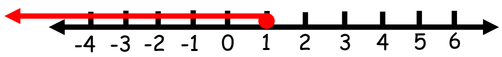
For the number line, we're looking for all the Integers that are bigger than -2.



Example 2: Graph the solution to $3(x - 2) \leq -3,\ x \in R$.

$3(x - 2) \leq -3$
$3x - 6 \leq -3$
$3x \leq -3 + 6$ (adding 6 to both sides)
$3x \leq 3$
$\frac{3x}{3} \leq \frac{3}{3}$ (dividing both sides by 3)
=> $x \leq 1$

For the number line, we're looking for all the Real numbers that are smaller than or equal to 1.



b) Double Inequalities:

Method 1: (Break inequality up into two inequalities)

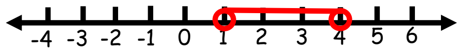
Example: Graph the solution to $-4 < 3x - 7 < 5,\ x \in R$.

We break up the inequality into two inequalities as indicated by the red and blue below.

$-4 < 3x - 7 < 5$

Inequality 1 (red)	Inequality 2 (blue)
$-4 < 3x - 7$	$3x - 7 < 5$
$-4 < 3x - 7$	$3x < 5 + 7$
$3 < 3x$	$3x < 12$
$1 < x$	$x < 4$

So, combining our two solutions, we want all the Real numbers that are bigger than 1 but less than 4. (not including the 1 and the 4)



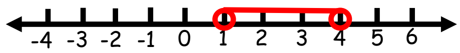
Method 2:

Tip:
Do the same thing to all three parts of the inequality to leave an 'x' in the middle.

Example: Graph the solution to $-4 < 3x - 7 < 5,\ x \in R$.

$-4 + 7 < 3x - 7 + 7 < 5 + 7,\ x \in R$ (+ 7 to eliminate -7 in the middle)

$3 < 3x < 12$
$\frac{3}{3} < \frac{3x}{3} < \frac{12}{3}$ (dividing all parts by 3)
$1 < x < 4$



c) Quadratic Inequalities:

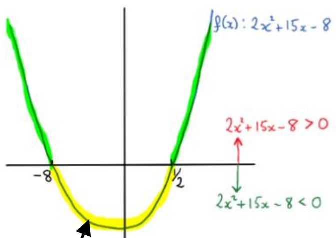
Steps for solving $ax^2 + bx + c < 0$:

1. Solve the equation $ax^2 + bx + c = 0$ to find the roots
2. If 'a' is positive => ∪ shape, if 'a' is negative => ∩ shape.
3. Use the above to sketch the graph of the function $ax^2 + bx + c$
4. Use the graph to solve the inequality.

Example: Solve the inequality $2x^2 + 15x - 8 \leq 0, x \in R$.

$2x^2 + 15x - 8 = 0$ (Step 1)
$(2x - 1)(x + 8) = 0$
$2x - 1 = 0 \quad or \quad x + 8 = 0$
$2x = 1 \quad or \quad x = -8$
$x = \frac{1}{2}$

- a = +2 => graph is ∪ shape



- The part of the graph we are interested in is below the x-axis (highlighted in yellow above). It is described by x values between -8 and ½. i.e. $-8 \leq x \leq \frac{1}{2}$

d) Rational Inequalities:

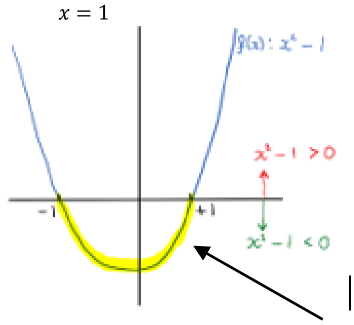
Example: Solve the inequality $\frac{2x + 4}{x + 1} < 3, x \in R$.

- We multiply both sides by $(x + 1)^2$:

$(x + 1)^2 \frac{2x + 4}{x + 1} < 3(x + 1)^2$
$(x + 1)(2x + 4) < 3(x + 1)^2$
$2x^2 + 6x + 4 < 3(x^2 + 2x + 1)$
$2x^2 + 6x + 4 < 3x^2 + 6x + 3$
$-x^2 + 1 < 0$
$x^2 - 1 < 0$

- We now solve the inequality above (See Section 6c):

$x^2 - 1 < 0$
$(x + 1)(x - 1) = 0$
$x + 1 = 0 \quad or \quad x - 1 = 0$
$x = -1 \quad or \quad x = 1$



Again, the section we're interested in is below the x-axis (highlighted yellow) so our solution is $-1 < x < 1$.

11) Modulus:

a) Modulus:

Notes:

- The **modulus** of a number x is the positive value of the number, without regard to its sign.
- Symbol: |x|
- So |3| = 3 and |-3| is also = 3.
- Our definition of modulus gives us a useful rule when dealing with moduli:

If |x| = a, then x = a or x = - a.

- When solving equations involving moduli a general rule of thumb is to square both sides similar to the way we solved surd equations.

Example: Solve the equation |2x – 5| = 7.

Method 1: Square both sides to eliminate the modulus symbol:

$(|2x - 5|)^2 = (7)^2$

$(2x - 5)^2 = 49$

$4x^2 - 20x - 24 = 0$

$x^2 - 5x - 6 = 0$

$(x - 6)(x + 1) = 0$

x = 6 or x = -1

Method 2: Use the definition of modulus and our rule above:

If |x| = a, then x = a or x = - a

- In this case |2x – 5| = 7, then

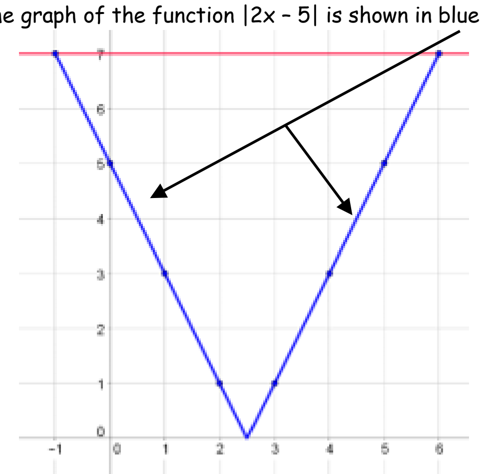
2x – 5 = 7	OR 2x – 5 = - 7
2x = 12	2x = - 2
x = 6	x = - 1

Method 3: Graphical Method

- To use this method, we need to have a look at what the graph of the modulus function |2x - 5| looks like.
- Let's try filling in some values for x and see what it looks like:

x values	function	y values
-1	|2(-1) – 5|	7
0	|2(0) – 5|	5
1	|2(1) – 5|	3
2	|2(2) – 5|	1
3	|2(3) – 5|	1
4	|2(4) – 5|	3
5	|2(5) – 5|	5
6	|2(6) – 5|	7

- The graph of the function |2x – 5| is shown in blue below.



- To find our solutions we can find where the graph of |2x – 5| = 7 using the graph i.e. x = -1 and x = 6

b) Modulus Inequalities:

Notes:

- When dealing with modulus inequalities, we can use a graph of modulus functions to establish another set of useful rules:

If |x| < a, then - a < x < a.
If |x| > a, then x < - a or x > a.

Example: Solve the inequality $|x - 4| \geq 3, x \in R$.

Method 1: We can square both sides:

$(|x - 4|)^2 \geq (3)^2$

$(x - 4)^2 \geq 9$

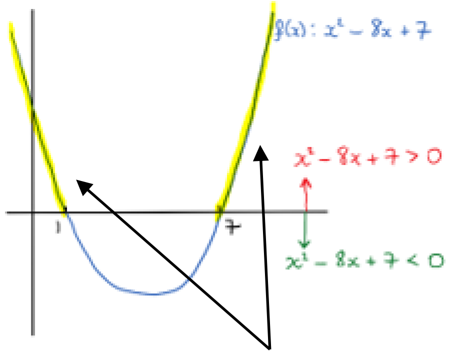
$x^2 - 8x + 16 \geq 9$

$x^2 - 8x + 7 \geq 0$

$(x - 7)(x - 1) = 0$

$x - 7 = 0 \quad or \quad x - 1 = 0$

$x = 7 \quad or \quad x = 1$



- So, our solution from the yellow part of the graph is $x \leq 1 \; or \; x \geq 7$.

Method 2: Using the rule above:

- If |x| > a, then x < - a or x > a.

=> If $|x - 4| \geq 3$, then

$x - 4 \leq -3 \quad or \quad x - 4 \geq 3$

$x \leq 1 \quad or \quad x \geq 7$

12) Inequality Proofs/Discriminants:

a) Inequality Proofs:

- For these proofs, the following is always true:

(Any real number)$^2 \geq 0$
-(Any real number)$^2 < 0$

Example: Prove that $a^2 - 10a + 25 + 4b^2 \geq 0$ for all a, b $\in R$.

- Our aim here will be to tidy up the left-hand side into (something)^2, which we can say is always ≥ 0

$a^2 - 10a + 25 + 4b^2$
$(a - 5)(a - 5) + (2b)^2$
$(a - 5)^2 + (2b)^2$ which is always positive for real numbers a and b
=> $a^2 - 10a + 25 + 4b^2 \geq 0$

b) Discriminants:

Notes:

- When solving Quadratic Equations, we use the 'b' Formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- The $b^2 - 4ac$ part of this formula is known as the **discriminant**.

If $b^2 - 4ac \geq 0$ => 2 distinct real roots.
If $b^2 - 4ac = 0$ => 2 equal roots.
If $b^2 - 4ac < 0$ => no real roots.

Example: Find the value of p in the equation $x^2 + 10x + p = 0$ if it has two equal roots.

- There are two equal roots so $b^2 - 4ac = 0$.
- In this case, a = 1, b = 10 and c = p

=> $b^2 - 4ac = (10)^2 - 4(1)(p)$
$= 100 - 4p = 0$
=> $100 = 4p$
=> $p = 25$

13) Indices:

a) The Laws of Indices:

See Tables pg 21

1) $a^p \times a^q = a^{p+q}$ e.g. $4^4 \times 4^3 = 4^7$

2) $\frac{a^p}{a^q} = a^{p-q}$ e.g. $\frac{5^3}{5^2} = 5^{3-2} = 5^1$

3) $(a^p)^q = a^{pq}$ e.g. $(5^2)^3 = 5^6$

4) $a^0 = 1$ e.g. $7^0 = 1$ or $(0.5)^0 = 1$

5) $a^{-p} = \frac{1}{a^p}$ e.g. $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

6) $(ab)^p = a^p b^p$ e.g. $(3x)^2 = 3^2x^2 = 9x^2$

7) $(\frac{a}{b})^p = \frac{a^p}{b^p}$ e.g. $(\frac{2}{3})^3 = \frac{2^3}{3^3} = \frac{8}{27}$

8) $a^{\frac{1}{2}} = \sqrt{a}$ e.g. $9^{\frac{1}{2}} = \sqrt{9} = 3$

$a^{\frac{1}{3}} = \sqrt[3]{a}$ e.g. $27^{\frac{1}{3}} = \sqrt[3]{27} = 3$

c) Table of the most Powers:

×	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8
2	2	4	8	16	32	64	128	256
3	3	9	27	81	243			
4	4	16	64	256				
5	5	25	125	625				
6	6	36	216					
7	7	49	343					
8	8	64	512					
9	9	81	729					
10	10	100	1000					

b) Solving equations with indices:

Steps:

1. Try and spot which powers you're dealing with, using the table below e.g. if you see a 9 and a 27 in the question, it would be powers of 3
2. Tidy up both sides of the equation into a single power using the laws of indices above. e.g. $5^x = 5^y$
3. If the bases are the same on both sides, you can now let the powers be equal to each other. i.e. x = y
4. Solve the simple equation to find your solution.

Example 1: Solve $3^x = 27\sqrt{3}$.

$3^x = 3^3.3^{\frac{1}{2}}$ ......using Law 8 above on the $\sqrt{3}$
$3^x = 3^{3+1/2}$.........using Law 1
$3^x = 3^{7/2}$ ....tidying up the power into a single fraction
=> $x = 7/2$ ....as the bases are equal

Example 2: Solve the equation

$2^{2x+1} - 5(2^x) + 2 = 0$

- Start by breaking up the power of the first term, using the Laws of Indices:

=> $2^{2x}2^1 - 5(2^x) + 2 = 0$ (Using Law 1)
=> $(2^x)^2 2^1 - 5(2^x) + 2 = 0$ (Using Law 3)
=> $2(2^x)^2 - 5(2^x) + 2 = 0$ (Moving the 2^1 out in front)

- Now let $y = 2^x$:

=> $2(y)^2 - 5(y) + 2 = 0$
=> $2y^2 - 5y + 2 = 0$
=> $(2y - 1)(y - 2) = 0$
=> $y = \frac{1}{2}$ or $y = 2$

- Now go back to our substitution to solve for the two x values:

If $y = \frac{1}{2}$ => $2^x = \frac{1}{2^1}$ => $2^x = 2^{-1}$ => $x = -1$	If $y = 2$ => $2^x = 2^1$ => $x = 1$

14) Logarithms:

a) Definition of Logs:

Notes:

- **Logarithms** are the opposite of raising a number to a particular power.
- The definition of logs is:

$$a^m = n \Leftrightarrow \log_a n = m$$

Example: If $2^5 = 32$, then we can write that using logs by writing: $\log_2 32 = 5$.

b) The Laws of Logs:

1) $\log_a(xy) = \log_a x + \log_a y$ $e.g.\ \log_2(20) = \log_2 5 + \log_2 4$

2) $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$ $e.g\ \log_5\left(\frac{2}{3}\right) = \log_5 2 - \log_5 3$

3) $\log_a(x^p) = p\log_a x$ $e.g\ \log_2 x^3 = 3\log_2 x$

4) $\log_a 1 = 0$ $e.g.\ \log_4 1 = 0$

5) $\log_a\left(\frac{1}{x}\right) = -\log_a x$ $e.g.\ \log_2\left(\frac{1}{8}\right) = -\log_2 8$

6) $\log_a(a^x) = x$ $e.g.\ \log_4(4^x) = x$

7) $a^{\log_a x} = x$ $e.g.\ 2^{\log_2 x} = x$

8) $\log_b x = \frac{\log_a x}{\log_a b}$ $e.g.\ \log_8 x = \frac{\log_2 x}{\log_2 8}$

See Tables pg 21

c) Simplifying Expressions:

Example: Simplify $\log_3 2 + 2\log_3 3 - \log_3 18$.

$\log_3 2 + 2\log_3 3 - \log_3 18$
$= \log_3 2 + 2(1) - [\log_3 9 + \log_3 2]$ (Laws 1, 6)
$= \log_3 2 + 2 - \log_3 9 - \log_3 2$
$= 2 - \log_3 9$ (as $\log_3 2 - \log_3 2 = 0$)
$= 2 - 2$ (as $\log_3 9 = 2$)
$= 0$

d) Solving Log Equations:

Notes:

- When solving equations with logs, use:

If $\log_a b = \log_a c$
=> b = c.

- Solutions of log equations always have to be checked.

Example: Solve the equation $\log_3(x+1) - \log_3(x-1) = 1$.

- Aim is to tidy up both sides into a single log, and use rule above.
- Use Law 2 to tidy up the LHS, and Law 6 to rewrite '1' on the RHS:

$\log_3(x+1) - \log_3(x-1) = 1$
=> $\log_3 \frac{x+1}{x-1} = \log_3 3$
=> $\frac{x+1}{x-1} = 3$
=> $x + 1 = 3x - 3$
=> $2x = 4$
=> $x = 2$

Check:
$\log_3(x+1) - \log_3(x-1) = 1$
=> $\log_3(2+1) - \log_3(2-1) = 1$
=> $\log_3(3) - \log_3(1) = 1$
=> $1 - 0 = 1$
=> $1 = 1$

e) Equations involving Change of Base Law:

Notes:

- Easier to change higher bases to lower bases

Example: Solve the equation

$\log_2(x+1) + \log_4(x) = \log_4(x^3+1)$.

- Change all logs to base 2 first:

=> $\log_2(x+1) + \frac{\log_2(x)}{\log_2 4} = \frac{\log_2(x^3+1)}{\log_2 4}$
=> $\log_2(x+1) + \frac{\log_2(x)}{2} = \frac{\log_2(x^3+1)}{2}$
=> $2\log_2(x+1) + \log_2(x) = \log_2(x^3 + 1)$

- RHS is a single log as it is, but we have a few more steps to do to the LHS to tidy it up into a single log:

=> $\log_2(x+1)^2 + \log_2(x) = \log_2(x^3 + 1)$
=> $\log_2(x+1)^2(x) = \log_2(x^3 + 1)$

- Can now eliminate the logs, as we have a single log on both sides, with the same power:

=> $(x)(x+1)^2 = x^3 + 1$
=> $(x)(x^2 + 2x + 1) = x^3 + 1$
=> $x^3 + 2x^2 + x = x^3 + 1$
=> $2x^2 + x - 1 = 0$
=> $x = \frac{1}{2}$ $or\ x = -1$

- Checking the answers eliminates $x = \frac{1}{2}$ as a solution, so the answer is x = -1

e) Problem Solving with Logs:

Example 1: Population, P, is modelled as $P = 12300(e^{0.073t})$, where t is in years. After how many years, will the population reach 20,000 people?

$P = 12300(e^{0.073t})$
=> $20000 = 12300(e^{0.073t})$
=> $e^{0.073t} = \frac{20000}{12300} = 1.626$ (dividing both sides by 12300)
=> $\ln e^{0.073t} = \ln 1.626$ (taking natural log of both sides to eliminate e)
=> 0.073t $= \ln 1.626$ (using Law 6 as $\ln e = 1$)
=> t $= \frac{\ln 1.626}{0.073} = \frac{0.486}{0.073} = 6.7yrs$

Example 2: Amount of radioactive tracer remaining after t days is given by $A = A_o(e^{-0.058t})$. A_o = starting amount. How many days, will it take for one half of the original amount to decay?

- Key in this type of question is we want **one half of the original amount**

=> $A = \frac{A_o}{2}$ or $A_o = 2A$

$A = A_o(e^{-0.058t})$
$A = 2A(e^{-0.058t})$ (filling in $A_o = 2A$)
=> $1 = 2(e^{-0.058t})$ (dividing both sides by A)
=> $0.5 = e^{-0.058t}$ (dividing both sides by 2)
=> $\ln 0.5 = \ln e^{-0.058t}$ (taking ln of both sides)
=> $\ln 0.5 = -0.058t$ (using Law 6 as $\ln e = 1$)
=> $t = \frac{\ln 0.5}{-0.058} = 11.95 = 12days$ (dividing both sides by -0.058)

15) Surds:

a) Surds:

Notes:

- A **surd** is a number in the form $\sqrt{\ }$ that **can't be written** as a **rational** number i.e. in the form $\frac{a}{b}$
  E.g. $\sqrt{2}$ and $\sqrt{3}$ are both surds but $\sqrt{9}$ is not as it can be written as $\frac{3}{1}$
- We can add/subtract similar surds together
  E.g. i) $3\sqrt{2} + 5\sqrt{2} = 8\sqrt{2}$
  ii) $4\sqrt{3} + 2\sqrt{2}$ ....we can't add these together as the $\sqrt{\ }$ parts are different

b) Reducing Surds:

- We can use the rule $\sqrt{ab} = \sqrt{a}\sqrt{b}$ to reduce larger surds into a simpler form:

Example: Simplify $\sqrt{50} + \sqrt{32}$

- We use 50 = 25 x 2 rather than 10 x 5 as 25 is a square number)

$\sqrt{50} + \sqrt{32}$
$= \sqrt{25}\sqrt{2} + \sqrt{16}\sqrt{2}$
$= 5\sqrt{2} + 4\sqrt{2}$
$= 9\sqrt{2}$

c) Solving Surd Equations:

Notes:

- A general rule of thumb for solving equations with surds is to **square both sides**.
- When squared both sides, there is sometimes an incorrect answer introduced so we always have to check our answers when solving surd equations.

Example 1: Solve the equation $\sqrt{2x+3} = 3$.

$(\sqrt{2x+3})^2 = (3)^2$ (Square both sides)
$2x + 3 = 9$
=> $x = 3$

Check:
$\sqrt{2x+3} = 3$
$\sqrt{2(3)+3} = 3$
$\sqrt{9} = 3$

Example 2: Solve the equation $\sqrt{x+7} + \sqrt{x+2} = 5$.

- More than one surd => rearrange the equation so there is one surd on each side and then square both sides as before:

$\sqrt{x+7} = 5 - \sqrt{x+2}$
$(\sqrt{x+7})^2 = (5 - \sqrt{x+2})^2$
$x + 7 = 25 - 10\sqrt{x+2} + x + 2$
$x + 7 = x + 27 - 10\sqrt{x+2}$

- Bring surd to one side and everything else to the other side:

$10\sqrt{x+2} = 20$

- Square both sides again as before:

$(10\sqrt{x+2})^2 = (20)^2$
$100(x+2) = 400$
$100x = 200$
$x = 2$

Check:
$\sqrt{x+7} + \sqrt{x+2} = 5$
$\sqrt{(2)+7} + \sqrt{(2)+2} = 5$
$\sqrt{9} + \sqrt{4} = 5$
$3 + 2 = 5$
$5 = 5$

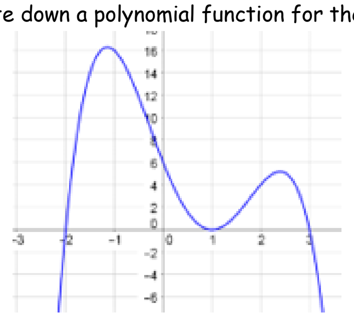
16) Polynomials/Factor Theorem/Solving Cubic Equations:

a) Polynomials:

Tips for identifying polynomials:

1. How many times does it touch or cross the x-axis? This gives degree.
2. Is degree odd or even? Odd => one arm up and one points down
3. Is right arm up or down? If up => positive leading coefficient
4. Let x = all the roots. Write out factors, and multiply together.
5. Sketch polynomial.

Example: Write down a polynomial function for the following:



- Both arms pointing down so degree must be even i.e. 4, 6, 8.....
- Right arm pointing down so leading coefficient must be negative
- If we count the number of times the graph crosses or touches the x-axis we can see it crosses twice and touches once => degree = 4 => there must be 4 roots
- Crosses x-axis at x = -2 and 3 and touches at x = 1
  => x = 1, x = 1, x = -2 and x = 3
  => Factors: (x – 1), (x – 1), (x + 2) and (x – 3)
- So, we can now put together our factors to get a possible polynomial for this graph:

$-(x-1)(x-1)(x+2)(x-3)$ or $-(x-1)^2(x+2)(x-3)$

b) Factor Theorem/Solving Cubic Equations:

Notes:

- This rule holds in general for all polynomials and is known as the Factor Theorem

A polynomial has a factor (x – a) if f(a) = 0.

Example: Solve the equation $x^3 - 2x^2 - 5x + 6 = 0$.

- Find a root. (must be a factor of +6 i.e. $\pm1, \pm2, \pm3, \pm6$
- We work through them in order:

$f(x) = x^3 - 2x^2 - 5x + 6$
$f(1) = (1)^3 - 2(1)^2 - 5(1) + 6 = 0$

- If x = 1 fails, keep going and try -1, 2, -2, 3, -3, -6 and 6.
- From Factor Theorem, if f(1) = 0, then x – 1 must be a factor.
- Find other factors using Long Division: (x – 3) and (x + 2).
- So, we can now factorise our original equation as:

$x^3 - 2x^2 - 5x + 6 = 0$
$(x-1)(x-3)(x+2) = 0$
=> x = 1 x = 3 x = -2

So, the graph of the function looks like?