Algebra

Q1. $(a^2-b^2)^{\frac{1}{2}}(a+b)^{-\frac{1}{2}}(a-b)^{\frac{3}{2}}$

$=(a-b)^{\frac{1}{2}}(a+b)^{\frac{1}{2}}(a+b)^{-\frac{1}{2}}(a-b)^{\frac{3}{2}}$ (Diff of 2 Sqs)

$=(a-b)^{\frac{1}{2}+\frac{3}{2}}(a+b)^{\frac{1}{2}-\frac{1}{2}}$ (Indices Law 1)

$=(a-b)^2(a+b)^0$

$=\boxed{(a-b)^2}$

Q2 Need to find a factor:

$f(1): 2(1)^3-3(1)^2-8(1)-3 \neq 0$

$f(-1): 2(-1)^3-3(-1)^2-8(-1)-3 = 0$

$\Rightarrow x+1$ is a factor

$$\begin{array}{r} 2x^2-5x-3 \\ x+1 \overline{)\,2x^3-3x^2-8x-3} \\ \underline{(-)\ 2x^3+2x^2} \\ -5x^2-8x \\ \underline{(+)\ -5x^2-5x} \\ -3x-3 \\ \underline{(+)\ 3x+3} \\ 0 \end{array}$$

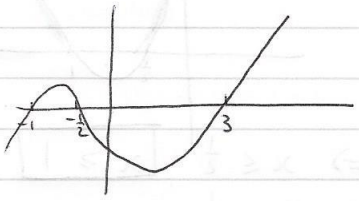
$\Rightarrow 2x^3-3x^2-8x-3=0$

$(x+1)(2x^2-5x-3)=0$

$(x+1)(2x+1)(x-3)=0$

$\boxed{x=-1}$ $\boxed{x=-\frac{1}{2}}$ $\boxed{x=3}$

Sketch: $+x^3 \Rightarrow$ Tail up



Q3. $\sqrt{3x+1}=\sqrt{x-1}+2$

Sq both sides:

$3x+1 = x-1+4+4\sqrt{x-1}$

$2x-2 = 4\sqrt{x-1}$

Sq both sides:

$4x^2-8x+4 = 16x-16$

$4x^2-24x+20=0$

$x^2-6x+5=0$

$(x-5)(x-1)=0$

$x=5 \quad x=1$

Check

$x=5$: $\sqrt{16}=\sqrt{4}+2$ ✓

$x=1$: $\sqrt{4}=\sqrt{0}+2$ ✓

Q4. $\frac{2-\sqrt{3}}{2+\sqrt{3}} \cdot \frac{2-\sqrt{3}}{2-\sqrt{3}}$

$= \frac{4+3-4\sqrt{3}}{(2)^2-(\sqrt{3})^2} = \frac{7-4\sqrt{3}}{1}$

$= \boxed{7-4\sqrt{3}}$

Q5. $4|x+1| = 3|x+1|$

$16(x^2+2x+1) = 9(x^2+2x+1)$

$16x^2+32x+16 = 9x^2+18x+9$

$7x^2+14x+7=0$

$x^2+2x+1=0$

$(x+1)(x+1)=0$

$\boxed{x=-1}$

Q6 $|2x-1| < 3$

Method 1: (Sq both sides)

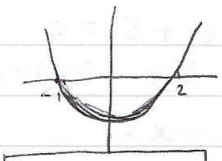
$4x^2 - 4x + 1 < 9$

$4x^2 - 4x - 8 < 0$

$x^2 - x - 2 < 0$

$(x-2)(x+1) = 0$

$x = 2 \quad x = -1$



$\Rightarrow \boxed{-1 < x < 2}$

Method 2: $|x| < a \Rightarrow -a < x < a$

$|2x-1| < 3$

$-3 < 2x - 1 < 3$

$-2 < 2x < 4$

$\boxed{-1 < x < 2}$

Q7 $3e^x - 7 + 2e^{-x} = 0$

$3e^x - 7 + \frac{2}{e^x} = 0$

Let $y = e^x$

$3y - 7 + \frac{2}{y} = 0$

$3y^2 - 7y + 2 = 0$

$(3y-1)(y-2) = 0$

$y = \frac{1}{3}$ or $y = 2$

$\Rightarrow e^x = \frac{1}{3}$ or $e^x = 2$

$\ln(e^x) = \ln(\frac{1}{3})$ $\quad \ln(e^x) = \ln 2$

$x = \ln(\frac{1}{3})$ $\quad x = \boxed{\ln 2}$

$= \boxed{-\ln 3}$

Q8. $\frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$

Sq both sides:

$\frac{a^2+2ab+b^2}{4} \leq \frac{a^2+b^2}{2} \quad (\times 4)$

$\Rightarrow a^2 + 2ab + b^2 \leq 2a^2 + 2b^2$

$\Rightarrow -a^2 + 2ab - b^2 \leq 0$

$a^2 - 2ab + b^2 \geq 0$

$(a-b)^2 \geq 0$ which is true

$\forall\ a, b \in \mathbb{R}$.

Q9. $\frac{x+3}{2x-1} \leq 4$

Mult both sides by $(2x-1)^2$:

$(2x-1)^2\left(\frac{x+3}{2x-1}\right) \leq 4(2x-1)^2$

$2x^2 + 6x - x - 3 \leq 4(4x^2 - 4x + 1)$

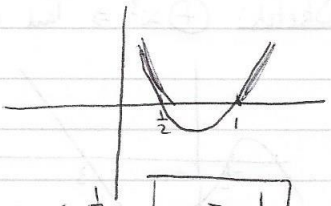
$2x^2 + 5x - 3 \leq 16x^2 - 16x + 4$

$-14x^2 + 21x - 7 \leq 0 \quad \div -7$

$2x^2 - 3x + 1 \geq 0$

$(2x-1)(x-1) = 0$

$x = \frac{1}{2} \quad x = 1$



$\Rightarrow x \leq \frac{1}{2}$ $\quad \boxed{x \geq 1}$

but $x \neq \frac{1}{2}$

$\Rightarrow \boxed{x < \frac{1}{2}}$

Q10 $x^2 + kx + (k+3) = 0$

Equal roots $\Rightarrow b^2 - 4ac = 0$

$k^2 - 4(1)(k+3) = 0$

$k^2 - 4k - 12 = 0$

$(k-6)(k+2) = 0$

$\boxed{k = 6}$ or $\boxed{k = -2}$

Q11. $\frac{3}{5}$ is a root $\Rightarrow f(\frac{3}{5}) = 0$

$5(\frac{3}{5})^3 + 7(\frac{3}{5})^2 - k(\frac{3}{5}) + 3 = 0$

$\frac{27}{25} + \frac{63}{25} - \frac{3k}{5} + 3 = 0$

$\frac{3k}{5} = \frac{33}{5}$

$\Rightarrow 3k = 33$

$\Rightarrow \boxed{k = 11}$

$x = \frac{3}{5}$

$\Rightarrow 5x = 3$

$5x - 3$ is a factor

$$\begin{array}{r} x^2 + 2x - 1 \\ 5x-3 \overline{)\ 5x^3 + 7x^2 - 11x + 3} \\ \underline{(-)\ 5x^3 - 3x^2} \\ 10x^2 - 11x \\ \underline{(-)\ 10x^2 - 6x} \\ -5x + 3 \\ \underline{(+)\ -5x + 3} \\ 0 \end{array}$$

$5x^3 + 7x^2 - 11x + 3 = 0$

$(5x-3)(x^2 + 2x - 1) = 0$

$x = \frac{-2 \pm \sqrt{4 - 4(-1)(1)}}{2}$

$= \frac{-2 \pm \sqrt{8}}{2}$

$= \boxed{-1 \pm \sqrt{2}}$

Q12 $2^{\frac{1}{4}} + 2^{\frac{1}{4}} + 2^{\frac{1}{4}} + 2^{\frac{1}{4}}$

$= 4 \cdot 2^{\frac{1}{4}}$

$= 2^2 \cdot 2^{\frac{1}{4}}$

$= 2^{2 + \frac{1}{4}} = \boxed{2^{\frac{9}{4}}}$

Q13. $6\log_x 2 + \log_2 x = 5$

Change of base:

$\log_2 x = \frac{\log_x x}{\log_x 2} = \frac{1}{\log_x 2}$

$\Rightarrow 6 \cdot \log_x 2 + \frac{1}{\log_x 2} = 5$

Let $y = \log_x 2$

$6y + \frac{1}{y} = 5$

$6y^2 + 1 = 5y$

$6y^2 - 5y + 1 = 0$

$(3y - 1)(2y - 1) = 0$

$y = \frac{1}{3}$ or $y = \frac{1}{2}$

$\log_x 2 = \frac{1}{3}$ $\quad$ $\log_x 2 = \frac{1}{2}$

$x^{\frac{1}{3}} = 2$ $\quad$ $x^{\frac{1}{2}} = 2$

$\boxed{x = 8}$ $\quad$ $\boxed{x = 4}$

Q14 $10x^2 + 4x + 1 = 2mx(2 - x)$

$10x^2 + 4x + 1 = 4mx - 2mx^2$

$(10 + 2m)x^2 + (4 - 4m)x + 1 = 0$

$b^2 - 4ac \geq 0$

$(4 - 4m)^2 - 4(10 + 2m)(1) \geq 0$

$16 - 32m + 16m^2 - 40 - 8m \geq 0$

$16m^2 - 40m - 24 \geq 0$

$2m^2 - 5m - 3 \geq 0$

$\boxed{m \leq -\frac{1}{2}}$ $\quad$ $\boxed{m \geq 3}$

Q15. $2^x + 2^{x+1} + 2^{x+2} = k.2^x$

$2^x + 2.2^x + 2^2.2^x = k.2^x$

$2^x + 2.2^x + 4.2^x = k.2^x$

$7.2^x = k.2^x$

$\Rightarrow \boxed{k = 7}$

Q16. $\log_2(7x+2) - \log_2(x+2) = 2$

$\log_2\left(\frac{7x+2}{x+2}\right) = 2$

$\Rightarrow 2^2 = \left(\frac{7x+2}{x+2}\right)$

$\Rightarrow \frac{4}{1} = \frac{7x+2}{x+2}$

$\Rightarrow 4(x+2) = 1(7x+2)$

$4x + 8 = 7x + 2$

$6 = 3x$

$\Rightarrow \boxed{x = 2}$

Q17. $y = a^2 \quad a^3 b = 1$

$\Rightarrow a = \sqrt{y}$

$= y^{\frac{1}{2}}$

$\Rightarrow a^3 = \left(y^{\frac{1}{2}}\right)^3$

$= y^{3/2}$

$\Rightarrow y^{3/2} . b = 1$

$y^{3/2} = \frac{1}{b}$

Sq both sides:

$y^3 = \frac{1}{b^2}$

Cube root both sides

$y = \sqrt[3]{\frac{1}{b^2}}$

$= \frac{\sqrt[3]{1}}{\sqrt[3]{b^2}}$

$= \frac{1}{b^{2/3}}$

$= \boxed{b^{-2/3}}$

Q18. $2^{2y+1} - 5.(2^y) + 2 = 0$

$2^{2y}.2^1 - 5(2^y) + 2 = 0$

$2.(2^y)^2 - 5(2^y) + 2 = 0$

Let $x = 2^y$

$2x^2 - 5x + 2 = 0$

$(2x - 1)(x - 2) = 0$

$x = \frac{1}{2} \quad x = 2$

$2^y = \frac{1}{2}$ or $2^y = 2^1$

$2^y = 2^{-1} \Rightarrow \boxed{y = 1}$

$\boxed{y = -1}$

Check: $y = 1$: $2^3 - 5.(2) + 2 = 0$ ✓

$y = -1$ $2^{-1} - 5(2^{-1}) + 2 = 0$ ✓

Q19. $L: 2x - 3y = 1$

$C: x^2 + xy - 4y^2 = 2$

Using L

$2x = 1 + 3y$

$x = \frac{1+3y}{2}$ (*)

$\Rightarrow C: \left(\frac{1+3y}{2}\right)^2 + \left(\frac{1+3y}{2}\right)(y) - 4y^2 = 2$

$\frac{1+6y+9y^2}{4} + \frac{y+3y^2}{2} - 4y^2 = 2$

$1 + 6y + 9y^2 + 2y + 6y^2 - 16y^2 = 8$

$-y^2 + 8y - 7 = 0$

$y^2 - 8y + 7 = 0$

$(y - 7)(y - 1) = 0$

$y = 7 \quad y = 1$

Using (*)

$y = 7 \Rightarrow x = \frac{1+3(7)}{2} = 11$

$y = 1 \Rightarrow x = \frac{1+3(1)}{2} = 2$

$\Rightarrow \boxed{(2, 1), (11, 7)}$

Q20. $32^{x-1} = 28$

$\log 32^{x-1} = \log 28$

$(x-1)\cdot \log 32 = \log 28$

$x - 1 = \dfrac{\log 28}{\log 32}$

$x = \dfrac{\log 28}{\log 32} + 1$

$\boxed{x = 1.096}$

Q21. Method 1:

$x = 4 \qquad x = -\frac{2}{3}$

$x - 4 = 0 \qquad 3x = -2$

$3x + 2 = 0$

$\Rightarrow (x-4)(3x+2) = 0$

$\boxed{3x^2 - 10x - 8 = 0}$

Method 2:

$x^2 - x(\text{SUM OF ROOTS}) + (\text{PROD OF ROOTS}) = 0$

$x^2 - x\left(4 - \frac{2}{3}\right) + (4)\left(-\frac{2}{3}\right) = 0$

$x^2 - x\left(\frac{10}{3}\right) - \frac{8}{3} = 0$

$\boxed{3x^2 - 10x - 8 = 0}$

Q22

$$\frac{x^{\frac{3}{2}} - x^{-\frac{1}{2}}}{x^{\frac{1}{2}} - x^{-\frac{1}{2}}} \cdot \frac{x^{\frac{1}{2}}}{x^{\frac{1}{2}}}$$

$$= \frac{x^{\frac{3}{2}+\frac{1}{2}} - x^{-\frac{1}{2}+\frac{1}{2}}}{x^{\frac{1}{2}+\frac{1}{2}} - x^{-\frac{1}{2}+\frac{1}{2}}}$$

$$= \frac{x^2 - 1}{x - 1} = \frac{(x-1)(x+1)}{(x-1)}$$

$$= \boxed{x+1}$$

Q23. $P = 40000(1.03)^n$

i) 12 yrs $\Rightarrow n = 12$

$P = 40000(1.03)^{12}$

$= \boxed{57030}$

ii) Pop Doubled $\Rightarrow P = 80000$

$80000 = 40000(1.03)^n$

$2 = (1.03)^n \quad (\div 40000)$

$\log 2 = \log(1.03)^n$

$\log 2 = n \cdot \log(1.03)$

$n = \dfrac{\log 2}{\log 1.03} = 23.45$

$= \boxed{23.5 \text{ yrs}}$

Q24. $f(x) = 2(x+2)^3(x^2)(x-2)^2$

→ $x^3 \cdot x^2 \cdot x^2 = x^7 \Rightarrow$ Odd degree

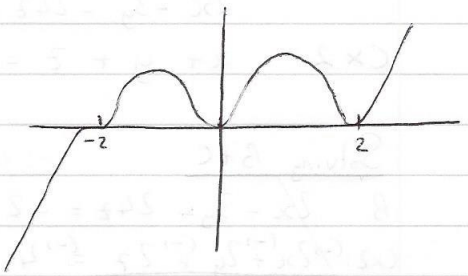
→ $(+)\, x^7 \Rightarrow$ Tail up on right

→ x^2 : double root @ $x = 0$

→ $(x+2)^3$: triple root @ $x = -2$

→ $(x-2)^2$: double root @ $x = 2$

→ 2 at front increases peaks & troughs



Q25. $(x+a)^2$ is factor

$\Rightarrow x^2+2ax+a^2$ is a factor

$$\begin{array}{r} x-2a \\ x^2+2ax+a^2 \overline{)\ x^3+0x^2+6px+k} \\ \underline{(-)\ x^3+2ax^2+a^2x} \\ -2ax^2+(6p-a^2)x+k \\ \underline{(+)\ -2ax^2-4a^2x-2a^3} \\ (6p+3a^2)x+k+2a^3 \end{array}$$

Rem $=0$

$\Rightarrow 6p+3a^2=0$ and $\boxed{k+2a^3=0}$

$6p=-3a^2$ $\quad 2a^3=-k$

$2p=-a^2$ $\quad a^3=\frac{-k}{2}$

$a^2=-2p$ $\quad a^6=\frac{k^2}{4}$

$a^6=-8p^3$

$\Rightarrow -8p^3=\frac{k^2}{4}$

$k^2=-32p^3$

$\boxed{k^2+32p^3=0}$

Q26. A × 40: $16x+5y+40z=100$

B × 6: $2x+2-3y-24z=0$

$2x-3y-24z=-2$

C × 2: $x+y+z=2$

Solving B + C

B: $2x-3y-24z=-2$

C×2: $(-)\ 2x+2y+2z=4$

$-5y-26z=-6$: D

Solving A + B

A: $16x+5y+40z=100$

B×8: $(-)\ 16x+24y+192z=16$

$29y+232z=116$: E

Solving D + E:

D×29: $-145y-754z=-174$

E×5: $145y+1160z=580$

$406z=406$

$\boxed{z=1}$

$\Rightarrow$ D: $-5y-26(1)=-6$

$-5y=-6+26$

$-5y=20$

$\boxed{y=-4}$

$\Rightarrow$ C: $x-4+1=2$

$\boxed{x=5}$

Q27.

i) $x^2-x(\text{sum of roots})+(\text{prod of roots})=0$

$x^2-x(3k)+(2k)(k)=0$

$\boxed{x^2-3kx+2k^2=0}$

ii) $x^2+ax+b=0$

1 root is twice the other

$\Rightarrow a=-3k \qquad b=2k^2$

$\Rightarrow k=\frac{a}{-3}$

$\Rightarrow b=2\left(\frac{a}{-3}\right)^2$

$b=2\left(\frac{a^2}{9}\right)$

$\boxed{9b=2a^2}$ Q.E.D.

Q28. Need an expression for $f(x)$ first:

Let P = original dose

After 1 hr: $f(x) = 0.85P$

After 2 hrs: $f(x) = (0.85)^2 . P$

After 3 hrs: $f(x) = (0.85)^3 . P$

$\Rightarrow$ After x hrs: $f(x) = P(0.85)^x$

i) After 4 hrs: $f(x) = 80(0.85)^4$

$= \boxed{41.76 \text{ mg}}$

ii) $80(0.85)^x = 20$

$(0.85)^x = 0.25 \quad (\div 80)$

$\log(0.85)^x = \log(0.25)$

$x \log(0.85) = \log(0.25)$

$x = \dfrac{\log(0.25)}{\log(0.85)} = \boxed{8.53}$

iii) $f(3) = (0.85)^3(80) = 49.13$

$f(2) = (0.85)^2(80) = 57.8$

$\Rightarrow$ Decreased by $57.8 - 49.13$

$= \boxed{8.67}$

Q29. i) Double root @ $-1 \Rightarrow (x+1)^2$

Single root @ $2 \Rightarrow (x-2)$

3 roots left $\Rightarrow$ Triple root @ $1 \Rightarrow (x-1)^3$

Both arms up $\Rightarrow \oplus x^6$

$\Rightarrow \boxed{(x+1)^2(x-2)(x-1)^3}$

ii) $f(x) = (x+1)^2(x-2)(x-1)^3$

$f(0) = (1)^2(-2)(-1)^3 = +2$

$\Rightarrow$ there is no multiplier @ the front of $f(x)$

$\Rightarrow f(3) = (4)^2(1)(2)^3 = \boxed{128}$