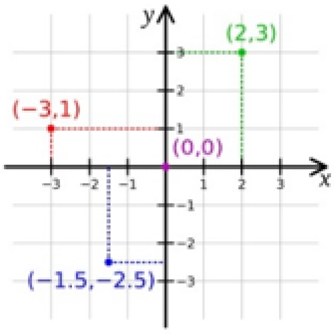
Topic 14: Coordinate Geometry of the Line

1) The Basics:

a) Cartesian Plane/Coordinates:

Notes:

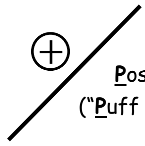
- Coordinates must be listed in brackets with a comma in between the two numbers
- We always list the X value first and the Y value second...see examples in diagram above.
- The point (0,0), shown in purple, is also called the **Origin**.
- The X and Y axes divides the plane up into 4 **quadrants**
  - Quadrant 1 is top right of the plane and they are numbered in an anti-clockwise direction



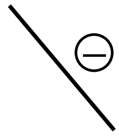
c) Slope:

Notes:

- Slope is a measure of the steepness of a line.
- Slopes can be negative or positive:

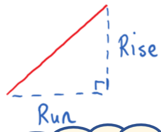


<u>P</u>ositive Slope
("<u>P</u>uff <u>P</u>uff <u>P</u>ositive")

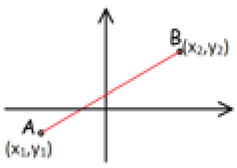


<u>N</u>egative Slope
("<u>N</u>ice <u>N</u>egative")

- There are three different ways we can find it:

Formula when we know **2 points**:	When given **diagram**:	When given the **equation of the line** in the form ax + by + c = 0
Slope |AB| $\frac{y_2 - y_1}{x_2 - x_1}$	 Slope = $\frac{RISE}{RUN}$	$\frac{-x\ number}{y\ number}$
Tables pg 18	**<u>Not</u>** in Tables	**<u>Not</u>** in Tables

b) Distance/Midpoint Formula:



Tables pg 18

Distance |AB|

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Midpoint of AB

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

d) Equation of a line:

Notes:

- A unique licence plate that identifies a particular line.
- To use the formula, we have to know:
  - **A point on the line**
  - The **slope** of the line (See section above)
- Once we know the two things above we use the formula:

$$y - y_1 = m(x - x_1)$$

Tables pg 18

- The equation of a line can also be given in the form:

$$y = mx + c$$

Tables pg 18

where **'m' = the slope and 'c' = the y-intercept** (where the line crosses the y-axis)

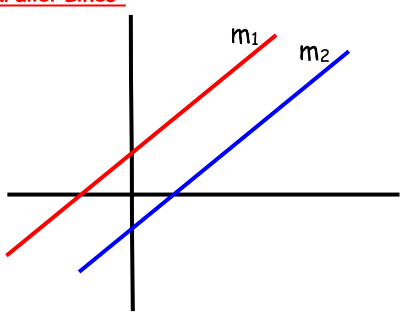
<u>Example:</u> A line with equation y = 3x – 5 has a slope of 3 and crosses the y-axis at the point (0, -5).

e) Intersecting Lines:

We can find where two lines meet by solving the equations simultaneously. See Algebra – Section 5a

2) Parallel/Perpendicular Lines:

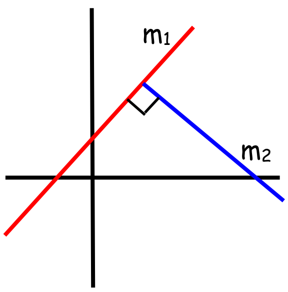
a) Parallel Lines:



Not in Tables

$$m_1 = m_2$$

b) Perpendicular Lines:



Not in Tables

$$m_1 \times m_2 = -1$$

N.B. If you have one slope, you can **'Flip & Change'** to get the other one

e.g. if $m_1 = 2/5$

=> $m_2 = -5/2$

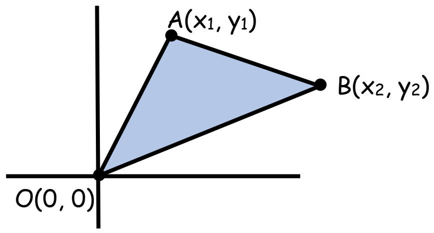
3) Area of a Triangle:

a) Triangle with one point at (0,0):

Note:

- To find the area of a triangle using the formula below, one of the points must be (0, 0).

Area = $\frac{1}{2}|x_1y_2 - x_2y_1|$ ← Tables pg 18



Example 1: Find the area of the triangle with coordinates (0,0), (4,-1) and (5,-3).

Area = $\frac{1}{2}|x_1y_2 - x_2y_1|$ (x_1, y_1) = (4, -1) (x_2, y_2) = (5, -3)

Area = $\frac{1}{2}|(4)(-3) - (5)(-1)|$

Area = $\frac{1}{2}|-12 + 5|$

Area = $\frac{1}{2}|-7|$ (taking the positive value of what's in the | |)

Area = $\frac{1}{2}(7) = 3.5$units²

b) Triangle with no points at (0,0):

Note:

- If none of the points are (0, 0), you have to move one point to (0, 0) and move the other points under the same translation.

Example 2: Find the area of the triangle with coordinates (3,-1), (5,2) and (-2,-3).

- Choose one point e.g. (3, -1) and move it to (0, 0) first and then move the other points by the same:
  - (3, -1) --------> (0, 0) (take 3 from x, add 1 to y)
  - (5, 2)---------> (2, 3) (take 3 from x, add 1 to y)
  - (-2, -3)-------> (-5, -2) (take 3 from x, add 1 to y)

- Now proceed as Example 1 with the three new points:

Area = $\frac{1}{2}|x_1y_2 - x_2y_1|$ (x_1, y_1) = (2, 3) (x_2, y_2) = (-5, -2)

Area = $\frac{1}{2}|(2)(-2) - (3)(-5)|$

Area = $\frac{1}{2}|-4 + 15|$

Area = $\frac{1}{2}|11|$

Area = $\frac{1}{2}(11) = 5.5$units²

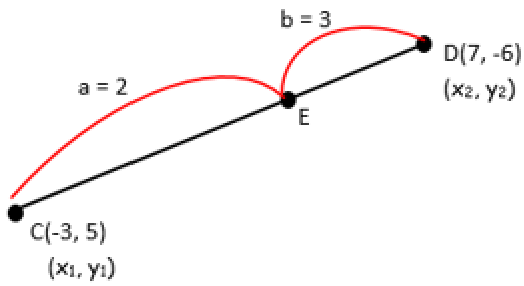
4) Line Segment Division/Graphing Lines:

a) Line Segment Division:

Notes:

- To find a point P along a line, that divides it into a certain ratio a:b :

P = $\left(\frac{bx_1 + ax_2}{b + a}, \frac{by_1 + ay_2}{b + a}\right)$ ← Tables pg 18



Example: C(-3, 5) and D(7, -6) are two points. Find the coordinates of the point E, when E is a point on [CD] such that |CE|:|DE| = 2:3.

P = $\left(\frac{bx_1 + ax_2}{b + a}, \frac{by_1 + ay_2}{b + a}\right)$

= $\left(\frac{3(-3) + 2(7)}{3 + 2}, \frac{3(5) + 2(-6)}{3 + 2}\right)$

= $\left(\frac{-9 + 14}{5}, \frac{15 - 12}{5}\right)$ = $\left(1, \frac{3}{5}\right)$

b) Graphing Lines:

Notes:

- We can use the equation of the line y = mx + c (See Section 1d) to draw/sketch lines.
- An alternative method to draw/sketch lines is to find the two points where the line crosses the X and Y axes.
- All points on the x-axis have a y-coordinate of 0.
- All points on the y-axis have a x-coordinate of 0.

Example: Graph the lines k: 3x – 4y + 12 = 0 and q: 3x + 2y – 5 = 0.

k: 3x - 4y + 12 = 0		q: 3x + 2y – 5 = 0	
x Intercept (y = 0)	y Intercept (x = 0)	x Intercept (y = 0)	y Intercept (x = 0)
3x - 4y + 12 = 0	3x - 4y + 12 = 0	3x + 2y – 5 = 0	3x + 2y – 5 = 0
3x – 0 + 12 = 0	0 – 4y + 12 = 0	3x + 0 – 5 = 0	0 + 2y – 5 = 0
3x = -12	-4y = -12	3x = 5	2y = 5
x = -4	y = 3	x = $\frac{5}{3}$	y = $\frac{5}{2}$
(-4, 0)	(0, 3)	$(\frac{5}{3}, 0)$	$(0, \frac{5}{2})$

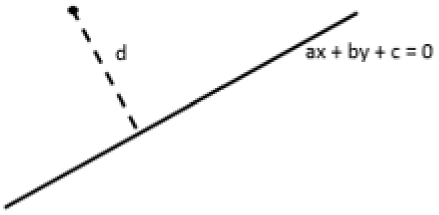
5) Perpendicular Distance from a point to a line:

Notes:

- The shortest distance from a point (x_1, y_1) to a line ax + by + c = 0 is the perpendicular distance and it's given by the formula:
- To find the shortest distance between two parallel lines, find a point on one line, and then use formula below to find distance from the point to the other line.
- **Parallel lines** will have equations ax + by + c = 0 and ax + by + d = 0.
- **Perpendicular lines** will have equations ax + by + c = 0 and bx – ay + c = 0 i.e. Swap x and y coefficients and change sign of y

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{(a)^2 + (b)^2}}$$

Tables pg 18



Example 1: Find the perpendicular distance from (-2, 5) to the line 3x – 2y + 7 = 0.

- The point (-2, 5) is our (x_1, y_1)
- We now fill into the formula:

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

$$= \frac{|(3)(-2) - (2)(5) + 7|}{\sqrt{(3)^2 + (-2)^2}}$$

$$= \frac{|-6 - 10 + 7|}{\sqrt{13}} = \frac{|-9|}{\sqrt{13}} = \frac{9}{\sqrt{13}}$$

Example 2: Find the shortest distance between the two parallel lines p: 2x – 3y + 4 = 0 and q: 2x – 3y – 8 = 0.

- Find a point on one of the lines first, so let's find a point on p.
- Find where the line crosses the x-axis i.e. where y = 0

  2x – 3y + 4 = 0
  2x – 3(0) + 4 = 0
  2x = -4
  x = -2 => Point = (-2, 0)

- Now use formula to find the shortest distance between the two lines i.e. distance from (-2, 0) to line q: 2x – 3y – 8 = 0

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

$$= \frac{|(2)(-2) - (3)(0) - 8|}{\sqrt{(2)^2 + (-3)^2}} = \frac{|-4 - 8|}{\sqrt{4 + 9}} = \frac{|-12|}{\sqrt{13}} = \frac{12}{\sqrt{13}}$$

Example 3: Find the equation of two lines that are parallel to the line x + 2y - 5 = 0 and a distance of $7\sqrt{5}$ from it.

- Any line parallel to the line x + 2y - 5 = 0 will be of the form x + 2y + c = 0.
- A point on the line x + 2y - 5 = 0 will be $7\sqrt{5}$ from these lines.
- A point on the line x + 2y - 5 = 0 would be (1, 2)

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

$$= \frac{|(1)(1) + (2)(2) + c|}{\sqrt{(1)^2 + (2)^2}} = \frac{|1 + 4 + c|}{\sqrt{1 + 4}} = \frac{|5 + c|}{\sqrt{5}}$$

- This distance is = $7\sqrt{5}$.

$$\frac{|5 + c|}{\sqrt{5}} = \frac{7\sqrt{5}}{1}$$

$|5 + c| = 7\sqrt{5}\sqrt{5}$ (Cross multiplying)

$|5 + c| = 35$

$|5 + c|)^2 = (35)^2$ (squaring both sides)

$25 + 10c + c^2 = 1225$

$c^2 + 10c - 1200 = 0$

=> $c = -40$ or $c = 30$

- So the two equations will be: $x + 2y - 40 = 0$ and $x + 2y + 30 = 0$.

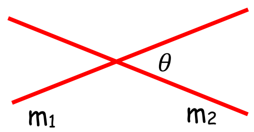
6) Angle between Lines:

Notes:

- Can find the angle θ between two lines if we know the slopes of the two lines using the formula:

$$\text{Tan } \theta = \pm \frac{m_1 - m_2}{1 + m_1 m_2}$$

Tables pg 18



- The $\pm$ gives us two values for Tan θ. One value will give us the acute angle between the lines and the other gives us the obtuse angle between the lines.

Example 1: Find the angle between the lines q: 3x – 4y + 8 = 0 and r: 2x + 5y – 3 = 0.

Slope of line q = $\frac{-x \text{ number}}{y \text{ number}} = \frac{-3}{-4} = \frac{3}{4}$ Slope of line r = $\frac{-2}{5}$

$$\text{Tan } \theta = \pm \frac{(\frac{3}{4}) - (-\frac{2}{5})}{1 + (\frac{3}{4})(-\frac{2}{5})}$$

$$\text{Tan } \theta = \pm \frac{\frac{3}{4} + \frac{2}{5}}{1 - \frac{3}{10}} = \pm \frac{\frac{23}{20}}{\frac{7}{10}} = \pm \frac{23}{14}$$

=> $\theta = \tan^{-1}(\frac{23}{14})$ = 58.67°

Obtuse angle
= 180 – 58.67
= 121.33°

Example 2: Find the equations of two lines through the point (-1, 2) that make an angle of 45° with the line 12x + 2y – 3 = 0.

- Firstly, we need to find the slope of the line we're given:

  12x + 2y – 3 = 0

  => Slope = $\frac{-x \text{ number}}{y \text{ number}} = \frac{-12}{2} = -6$

- Let the slope of the two lines we're looking for = 'm'

$$\text{Tan } \theta = \pm \frac{(-6) - (m)}{1 + (-6)(m)} = \pm \frac{-6 - m}{1 - 6m}$$

- The angle between the lines is 45° and Tan (45) = 1 so

$$1 = \pm \frac{-6 - m}{1 - 6m}$$

- Want the positive value of the RHS so, square both sides:

$$1 = \frac{36 + 12m + m^2}{1 - 12m + 36m^2}$$

$36 + 12m + m^2 = 1 - 12m + 36m^2$ (Cross multiplying)

$35m^2 - 24m - 35 = 0$

$(7m + 5)(5m - 7) = 0$

=> $m = -\frac{5}{7}$ or $m = \frac{7}{5}$

- So, the equation of our two lines through the point (-1, 2)

$y - y_1 = m(x - x_1)$ $y - 2 = \frac{-5}{7}(x + 1)$ $7(y - 2) = -5(x + 1)$ $5x + 7y - 9 = 0$	$y - y_1 = m(x - x_1)$ $y - 2 = \frac{7}{5}(x + 1)$ $5(y - 2) = 7(x + 1)$ $7x - 5y + 3 = 0$