Worked Solutions

Q1

$$\frac{dy}{dx} = 6y.\sin x$$

$$dy = 6y.\sin x.\,dx$$

$$\int \frac{1}{y}.\,dy = \int 6\sin x.\,dx$$

$$\ln y = 6(-\cos x) + c$$

$$\ln y = -6\cos x + c$$

$\underline{y = 1 \text{ when } x = \pi}$

$$\ln(1) = -6\cos(\pi) + c$$

$$0 = -6(-1) + c$$

$$c = -6$$

$$\Rightarrow \ln y = -6\cos x - 6$$

$$\Rightarrow \boxed{y = e^{-6(\cos x + 1)}}$$

Q2.

$$\frac{dy}{dx} = \sqrt{1-y^2}$$

$$dy = \sqrt{1-y^2}\,.dx$$

$$\int \frac{1}{\sqrt{1-y^2}}.\,dy = \int 1.\,dx$$

$$\sin^{-1}\left(\frac{y}{1}\right) = x + c$$

$$\sin^{-1}(y) = x + c$$

$\underline{y = 0 \text{ when } x = 1}$

$$0 = 1 + c$$

$$c = -1$$

$$\Rightarrow \sin^{-1}(y) = x - 1$$

$$\boxed{y = \sin(x-1)}$$

Q3

$$x \cdot \frac{dy}{dx} = y - xy$$

$$x \cdot dy = (y - xy) \cdot dx$$

$$x \cdot dy = y(1-x) \cdot dx$$

$$\frac{1}{y} \cdot dy = \left(\frac{1-x}{x}\right) \cdot dx$$

$$\int \frac{1}{y} \cdot dy = \int \left(\frac{1}{x} - 1\right) \cdot dx$$

$$\ln y = \ln x - x + c$$

$y = 3$ when $x = 5$

$$\ln 3 = \ln 5 - 5 + c$$

$$c = \ln 3 - \ln 5 + 5$$

$$= \ln(3/5) + 5$$

$$\Rightarrow \ln y = \ln x - x + \ln(3/5) + 5$$

$$\ln y - \ln x - \ln(3/5) = 5 - x$$

$$\ln(y/x) - \ln(3/5) = 5 - x$$

$$\frac{y}{x} \div \frac{3}{5} = \frac{y}{x} \times \frac{5}{3} = \frac{5y}{3x}$$

$$\ln(5y/3x) = 5 - x$$

$$\frac{5y}{3x} = e^{5-x}$$

$$5y = 3x \cdot e^{5-x}$$

$$\boxed{y = \frac{3}{5} x \cdot e^{5-x}}$$

Q4

$$3 \cdot \frac{dy}{dx} + \frac{x}{x^2+1} = 0$$

$$3 \cdot \frac{dy}{dx} = \frac{-x}{x^2+1}$$

$$\int 3 \cdot dy = \int \left(\frac{-x}{x^2+1}\right) \cdot dx$$

$u = x^2+1$

$du = 2x.dx$

$\frac{1}{2} du = x.dx$

$$\int 3.dy = -\frac{1}{2} \int \frac{1}{u} \cdot du$$

$$3y = -\frac{1}{2} \ln u + c$$

$$3y = -\frac{1}{2} \ln(x^2+1) + c$$

$y = 2$ when $x = 0$

$$3(2) = -\frac{1}{2} \ln((0)^2+1) + c$$

$$6 = -\frac{1}{2} \ln(1) + c$$

$$6 = -\frac{1}{2}(0) + c$$

$$c = 6$$

$$\Rightarrow 3y = -\frac{1}{2} \ln(x^2+1) + 6$$

$$\boxed{y = -\frac{1}{6} \ln(x^2+1) + 2}$$

Q5.

$$x \cdot \frac{dy}{dx} = (1+2x^2)y^2$$

$$x \cdot dy = (1+2x^2)y^2 \cdot dx$$

$$\frac{1}{y^2} \cdot dy = \frac{1+2x^2}{x} \cdot dx$$

$$\int y^{-2} \cdot dy = \int \left(\frac{1}{x} + 2x\right) \cdot dx$$

$$\frac{y^{-1}}{-1} = \ln x + 2\left(\frac{x^2}{2}\right)$$

$$-\frac{1}{y} = \ln x + x^2 + c$$

$y = 1$ when $x = 1$

$$-\frac{1}{1} = \ln(1) + (1)^2 + c$$

$$-1 = 1 + c$$

$$c = -2$$

$$\Rightarrow \quad -\frac{1}{y} = \ln x + x^2 - 2$$

$$\frac{1}{y} = \frac{2 - \ln x - x^2}{1}$$

$$\boxed{y = \frac{1}{2 - \ln x - x^2}}$$

Q6.

$$\frac{dy}{dx} + y^2 . \cos x = 0$$

$$\frac{dy}{dx} = -y^2 \cos x$$

$$dy = -y^2 . \cos x . \, dx$$

$$\frac{1}{y^2} . dy = -\cos x . \, dx$$

$$\int y^{-2} . dy = \int -\cos x . \, dx$$

$$\frac{y^{-1}}{-1} = -\sin x$$

$$-\frac{1}{y} = -\sin x$$

$$\frac{1}{y} = \sin x + C$$

$y = 2$ when $x = \frac{\pi}{6}$

$$\frac{1}{2} = \sin\left(\pi/6\right) + c$$

$$\frac{1}{2} = \frac{1}{2} + c$$

$$c = 0$$

$$\Rightarrow \frac{1}{y} = \frac{\sin x}{1}$$

$$\boxed{y = \frac{1}{\sin x}}$$

Q7.

$$200 \frac{dv}{dt} = 1600 + v^2$$

$$200\, dv = (1600 + v^2).dt$$

$$\int \frac{200}{1600+v^2}.dv = \int 1.\, dt$$

$$200 \int \frac{1}{1600+v^2}.dv = \int 1.\, dt$$

$$200\left(\frac{1}{40} \tan^{-1}\left(\frac{v}{40}\right)\right) = t + c$$

$$5 \tan^{-1}\left(\frac{v}{40}\right) = t + c$$

<u>v = 0 when t = 50</u>

$$5 \tan^{-1}(0) = 50 + c$$

$$c = -50$$

$$\Rightarrow 5 \tan^{-1}\left(\frac{v}{40}\right) = t - 50$$

$$\tan^{-1}\left(\frac{v}{40}\right) = \frac{t-50}{5}$$

$$\frac{v}{40} = \tan\left(\frac{t-50}{5}\right)$$

$$v = 40 \tan\left(\frac{t-50}{5}\right)$$

@ <u>t = 55</u>

$$v = 40 \tan\left(\frac{55-50}{5}\right)$$

$$= 40 \tan(1)$$ → * RADIAN Mode

$$= 40(1.5574)$$

$$= \boxed{62.296}$$

Q8.

$$\frac{dv}{dt} = g - kv$$

$$dv = (g-kv).dt$$

$$\int \frac{1}{g-kv}.dv = \int 1.dt$$

$$-\frac{1}{k}\ln(g-kv) = t + c$$

@ $t=0$ $v=0$

$$-\frac{1}{k}\ln(g-k(0)) = 0 + c$$

$$c = -\frac{1}{k}\ln g$$

$$-\frac{1}{k}\ln(g-kv) = t - \frac{1}{k}\ln g$$

$$\frac{1}{k}\ln g - \frac{1}{k}\ln(g-kv) = t$$

$$\ln g - \ln(g-kv) = kt$$

$$\ln\left(\frac{g}{g-kv}\right) = kt$$

$$\frac{g}{g-kv} = e^{kt}$$

$$g = (g-kv)e^{kt}$$

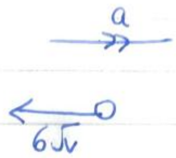
$$\frac{g}{e^{kt}} = g - kv$$

$$+kv = g - \frac{g}{e^{kt}}$$

$$kv = g - ge^{-kt}$$

$$\boxed{v = \frac{g - ge^{-kt}}{k}}$$

Q9.



$$F = ma$$

$$2a = -6\sqrt{v}$$

$$a = -3\sqrt{v}$$

$$\frac{dv}{dt} = -3\sqrt{v}$$

$$dv = -3\sqrt{v} \, . \, dt$$

$$\frac{1}{\sqrt{v}} . dv = -3 . dt$$

$$\int v^{-\frac{1}{2}} . dv = \int -3 . dt$$

$$\frac{v^{\frac{1}{2}}}{\frac{1}{2}} = -3t + c$$

$$2\sqrt{v} = -3t + c$$

@ $t = 0$ $v = 9$

$$2\sqrt{9} = -3(0) + c$$

$$c = 2\sqrt{9} = 6$$

$$\Rightarrow 2\sqrt{v} = -3t + 6$$

at rest $\Rightarrow v = 0$

$$2\sqrt{0} = -3t + 6$$

$$3t = 6$$

$$t = \boxed{2s}$$

Q10.

$$a = -(1+v)$$

$$\frac{dv}{dt} = -(1+v).$$

$$dv = -(1+v)\,.\,dt$$

$$\int \frac{1}{1+v}\,.\,dv = \int -1\,.\,dt$$

$$\ln(v+1) = -t + c$$

@ $t = 0 \quad v = 10$

$$\ln(10+1) = 0 + c$$

$$c = \ln(11)$$

$$\Rightarrow \ln(v+1) = -t + \ln(11)$$

$$\ln(v+1) - \ln(11) = -t$$

$$\ln\left(\frac{v+1}{11}\right) = -t$$

$$\frac{v+1}{11} = e^{-t}$$

$$v+1 = 11.e^{-t}$$

$$v = 11e^{-t} - 1$$

At rest $\Rightarrow$ $v = 0$

$$0 = 11e^{-t} - 1$$

$$11e^{-t} = 1$$

$$\ln(e^{-t}) = \ln\left(\frac{1}{11}\right)$$

$$-t = -2.3978$$

$$t = \boxed{2.4s}$$

Q11.

$a \propto -v$

$a = -kv$

$\frac{dv}{dt} = -kv$

$dv = -kv.\,dt$

$\int \frac{1}{v}.\,dv = \int -k.\,dt$

$\ln(v) = -kt + c$

@ $t = 0 \quad v = 20$

$\ln(20) = 0 + c$

$c = \ln(20)$

$\Rightarrow \ln(v) = -kt + \ln(20)$

@ $t = 5 \quad v = 10$

$\ln(10) = -k(5) + \ln(20)$

$\ln(10) - \ln(20) = -5k$

$\ln(\tfrac{1}{2}) = -5k$

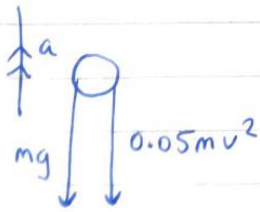
$k = \frac{\ln(\frac{1}{2})}{-5} = \frac{-\ln(2)}{-5} = \boxed{\frac{\ln(2)}{5}}$ or $\boxed{0.1386}$

Q12.

$$F = ma$$

$$-mg - 0.05mv^2 = ma$$

$$a = -g - 0.05v^2$$



$$v.\frac{dv}{dx} = -g - 0.05v^2$$

$$v.dv = (-g - 0.05v^2).dx$$

$$\int \frac{v}{0.05v^2 + g}.dv = \int -1.dx$$

$$u = 0.05v^2 + g$$
$$\frac{du}{dv} = 0.1v$$
$$du = 0.1v.dv$$

$$10\int \frac{1}{u}.du = \int -1.dx \qquad 10.du = v.dv$$

$$10\ln u = -x + c$$

$$10\ln(0.05v^2 + g) = -x + c$$

@ $x = 0 \quad v = 50$

$$10\ln(0.05(50)^2 + g) = -0 + c$$

$$c = 10\ln(125 + g)$$

$$10\ln(0.05v^2 + g) = -x + 10\ln(125 + g)$$

i) Max height $\Rightarrow v = 0$:

$$10\ln(0 + g) = -x + 10\ln(125 + g)$$

$$x = 10\ln(125 + g) - 10\ln(g)$$

$$= 10\ln\left(\frac{125 + g}{g}\right)$$

$$= \boxed{26.21\,m}$$

ii) $$\frac{dv}{dt} = -(g + 0.05v^2)$$

$$\int \frac{1}{g + 0.05v^2}.dv = \int -1.dt$$

$$\int \frac{20}{20g + v^2}.dv = \int -1.dt$$

$$\int \frac{20}{v^2 + 196}.dv = \int -1.dt$$

$$20 \int \frac{1}{v^2 + 14^2} \, dv = \int -1 \, dt$$

$$20 \left(\frac{1}{14}\right) \tan^{-1}\left(\frac{v}{14}\right) = -t + c$$

$$\frac{10}{7} \tan^{-1}\left(\frac{v}{14}\right) = -t + c$$

@ $t = 0 \quad v = 50$

$$\frac{10}{7} \tan^{-1}\left(\frac{50}{14}\right) = c$$

RAD Mode

$$c = 1.86$$

$$\frac{10}{7} \tan^{-1}\left(\frac{v}{14}\right) = -t + 1.86$$

Max height $\Rightarrow v = 0$

$$\frac{10}{7} \tan^{-1}(0) = -t + 1.86$$

$$t = \boxed{1.86s}$$

iii) Air resistance ignored $\Rightarrow$ uni acceleration

$$v^2 = u^2 + 2as$$

$$(0)^2 = (50)^2 + 2(-9.8)s$$

$$19.6s = 2500$$

$$s = \frac{2500}{19.6} = \boxed{127.55m}$$

$$v = u + at$$

$$0 = 50 - 9.8t$$

$$9.8t = 50$$

$$t = \boxed{5.1s}$$

Q13.

$P = T.v$

$\frac{42000}{v} = T$

$F = ma$

$\frac{42000}{v} = 600a$

$\Rightarrow a = \frac{7}{v}$

$\frac{dv}{dt} = \frac{7}{v}$

$dv = \frac{7}{v}.dt$

$\int v.dv = \int 7.dt$

$\frac{v^2}{2} = 7t + c$

@ $t = 0 \quad v = 10$

$\frac{(10)^2}{2} = 7(0) + c$

$c = 50$

$\frac{v^2}{2} = 7t + 50$

$v^2 = 14t + 100$

@ $v = 30$

$(30)^2 = 14t + 100$

$14t = 800$

$t = \boxed{\frac{400}{7} s}$ or $\boxed{57.14s}$

Q14.

$$\frac{dP}{dt} = kP$$

$$dP = k.P.dt$$

$$\int \frac{1}{P}.dP = \int k.dt$$

$$\ln P = kt + c$$

@ $t = 0$ $P = 150000$

$$\ln(150000) = c$$

$$\ln(P) = kt + \ln(150000)$$

$$\ln\left(\frac{P}{150000}\right) = kt$$

$$P = 150000 e^{kt}$$

@ $t = 4$ $P = 0.98(150000) = 147000$

$$\ln(147000) = k(4) + \ln(150000)$$

$$4k = \ln(147000) - \ln(150000)$$

$$4k = -0.0202$$

$$k = -0.00505$$

$$\Rightarrow P = 150000 e^{-0.00505t}$$

@ $t = 25$

$$P = 150000 e^{-0.00505(25)}$$

$$= \boxed{132,209}$$

Q15.

$$\frac{dI}{dt} = kI$$

$$\int \frac{1}{I}.dI = \int k.dt$$

$$\ln(I) = kt + c$$

@ $t = 0 \quad I = 600$

$$\ln(600) = c$$

$$\Rightarrow \ln(I) = kt + \ln(600)$$

@ $t = 10 \quad I = 300$

$$\ln(300) = 10k + \ln(600)$$

$$10k = \ln(300) - \ln(600)$$

$$10k = -0.693147$$

$$k = -0.0693$$

$$\Rightarrow \ln(I) = -0.0693t + \ln(600)$$

$$\ln(I/600) = -0.0693t$$

$$\frac{I}{600} = e^{-0.0693t}$$

$$I = 600e^{-0.0693t}$$

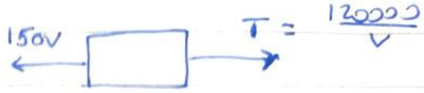
@ $t = 50$

$$I = 600e^{-0.0693(50)}$$

$$= \boxed{18.76g}$$

Q16.

i)



$P = Tv$

$120000 = Tv$

$\Rightarrow T = \dfrac{120000}{v}$

$F = ma$

$\dfrac{120000}{v} - 150v = 600a$

$\dfrac{200}{v} - \dfrac{v}{4} = a$

$a = \dfrac{800 - v^2}{4v}$

$\Rightarrow 4a = \dfrac{800 - v^2}{v}$

$\Rightarrow \boxed{4 \cdot \dfrac{dv}{dt} = \dfrac{800 - v^2}{v}}$

ii)

$4 \cdot dv = \dfrac{800 - v^2}{v} \cdot dt$

$\dfrac{4v}{800 - v^2} \cdot dv = 1 \cdot dt$

$u = 800 - v^2$

$du = -2v \cdot dv$

$-\frac{1}{2} \cdot du = v \cdot dv$

$\Rightarrow 4\left(-\frac{1}{2}\right) \int \frac{1}{u} \cdot du = \int 1 \cdot dt$

$-2 \ln u = t + c$

$-2 \ln(800 - v^2) = t + c$

@ $t = 0$ $v = 20$

$-2 \ln(800 - (20)^2) = c$

$c = -2 \ln 400$

$-2 \ln(800 - v^2) = t - 2\ln(400)$

@ $v = 25$ $t = 2\ln(400) - 2\ln(800 - (25)^2)$

$= 11.98 - 10.329 = \boxed{1.65 s}$

iii) $2\ln(400) - 2\ln(800 - v^2) = t$

$2\ln\left(\frac{400}{800 - v^2}\right) = t$

$\ln\left(\frac{400}{800 - v^2}\right) = \frac{t}{2}$

$\frac{400}{800 - v^2} = e^{t/2}$

$(800 - v^2)e^{t/2} = 400$

$800 - v^2 = 400e^{-t/2}$

$v^2 = 800 - 400e^{-t/2}$

$v = \sqrt{800 - 400e^{-t/2}}$

Max will be as $t \to \infty \Rightarrow$ which means $e^{-t/2} \to 0$

$\Rightarrow \quad v = \sqrt{800 - 400(0)}$

$= \sqrt{800}$

$= \boxed{20\sqrt{2} \text{ m/s}}$

SEC HL Sample Q3(a)

(a) A particle has initial displacement s_0 from a fixed point P. It moves away from P with initial velocity u and constant acceleration $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$.

Use calculus to derive an expression for s, the displacement of the particle from P at any time t.

$\frac{dv}{dt} = a$

=> $1.dv = a.dt$ (Multiply by dt)

=> $\int 1.dv = \int a.dt$

=> $v = at + C$

$v = u \text{ when } t = 0$ (Initial conditions)

=> $u = a(0) + C$

=> $u = C$

=> $v = at + u$

=> $v = u + at$

$v = \frac{ds}{dt}$

=> $\frac{ds}{dt} = u + at$

=> $ds = (u + at).dt$ (Multiply by dt)

=> $\int 1\, ds = \int (u + at).dt$

=> $s = ut + a\frac{t^2}{2} + C$

$s = s_0 \text{ when } t = 0$ (Initial conditions)

=> $s_0 = u(0) + a\frac{(0)^2}{2} + C$

=> $s_0 = C$

=> $s = ut + a\frac{t^2}{2} + s_0$

2024 Q6(b)

$v.dv = e^{\frac{v^2}{4}}.ds$

=> $\frac{v}{e^{\frac{v^2}{4}}}.dv = 1.ds$ (Dividing both sides by $e^{\frac{v^2}{4}}$)

=> $ve^{-\frac{v^2}{4}}.dv = 1.ds$ (Using law of indices to bring $e^{\frac{v^2}{4}}$ above the line)

Substitution: Let $u = \frac{v^2}{4}$

=> $du = \frac{2v}{4}.dv$

=> $v.dv = 2.du$

=> $\int e^{-u}.2.du = \int 1.ds$

=> $-2e^{-u} = s + C$

=> $-2e^{-\frac{v^2}{4}} = s + C$

Initial conditions: $v = 0$ when $s = 0$:

$-2e^{-\frac{(0)^2}{4}} = (0) + C$

=> $C = -2$

Subbing C back into integral:

=> $-2e^{-\frac{v^2}{4}} = s - 2$

=> $e^{-\frac{v^2}{4}} = \frac{2-s}{2}$ (Dividing both sides by -2)

=> $-\frac{v^2}{4} = \ln(\frac{2-s}{2})$ (Taking ln of both sides)

=> $-v^2 = 4\ln(\frac{2-s}{2})$

=> $v^2 = -4\ln(\frac{2-s}{2})$

=> $v^2 = 4\ln(\frac{2}{2-s})$ (Using law of logs $-\log b = \log\frac{1}{b}$)

=> $v = \sqrt{4\ln(\frac{2}{2-s})}$

(ii) Calculate the velocity of the particle when $s = 0.3$ m.

$$v = \sqrt{4\ln(\frac{2}{2-s})}$$

$$= \sqrt{4\ln(\frac{2}{2-0.3})} = \sqrt{0.650075} = 0.806\ m/s$$

2023 Deferred Q4

(i) Solve this simplified differential equation to find an expression for P in terms of t.

$$\frac{dP}{dt} = rP$$

=> $dP = 0.08P.dt$ (Multiplying both sides by dt)

=> $\frac{1}{P}.dP = 0.08.dt$ (Dividing both sides by P)

=> $\int \frac{1}{P}.dP = \int 0.08.dt$

=> $\ln P = 0.08t + C$

Initial conditions: $P = 20$ when $t = 0$:

$\ln 20 = (0.08)(0) + C$

=> $C = \ln(20)$

=> $\ln P = 0.08t + \ln(20)$ (Subbing C back into integral)

=> $\ln P - \ln(20) = 0.08t$

=> $\ln \frac{P}{20} = 0.08t$

=> $\frac{P}{20} = e^{0.08t}$ (Taking e of both sides)

=> $P = 20e^{0.08t}$

(ii) Calculate P to the nearest whole number when $t = 12$ weeks.

$$P = 20e^{0.08t}$$

=> $P = 20e^{0.08(12)} = 52.23 = 52\ weeks$

(iii) Explain why this approximation of Verhulst's model is not practical for predicting the long-term behaviour of the population of insects.

The population $P = 20e^{0.08t}$ will keep increasing as time goes on (as $t \to \infty$). It said above that the model was based on P being small relative to K, but if P keeps increasing, P will not stay small relative to K.

(iv) Solve the differential equation for Verhulst's model:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

to find an expression that relates P, K and t.

Note that $\frac{1}{y(x-y)} = \frac{1}{x}\left(\frac{1}{y} + \frac{1}{x-y}\right)$.

$\frac{dP}{dt} = 0.08P(1 - \frac{P}{K})$

=> $dP = 0.08P\left(1 - \frac{P}{K}\right).dt$ (Multiplying both sides by dt)

=> $\frac{1}{P(1-\frac{P}{K})}.dP = 0.08.dt$ (Dividing both sides by $P(1 - \frac{P}{K})$)

=> $\frac{K}{P(K-P)}.dP = 0.08.dt$ (Multiplying above and below by K)

Using the result given in the question above:

$\frac{K}{P(K-P)} = K.\frac{1}{K}\left(\frac{1}{P} + \frac{1}{K-P}\right) = \frac{1}{P} + \frac{1}{K-P}$

=> $(\frac{1}{P} + \frac{1}{K-P}).dP = 0.08.dt$

=> $\int\left(\frac{1}{P} + \frac{1}{K-P}\right).dP = \int 0.08.dt$

=> $\ln P + \ln(K-P)(-1) = 0.08t + C$

=> $\ln P - \ln(K-P) = 0.08t + C$

=> $\ln\frac{P}{K-P} = 0.08t + C$ (Using Law of Logs: $\ln a - \ln b = \ln\frac{a}{b}$)

Initial conditions: $P = 20$ when $t = 0$:

$\ln\frac{20}{K - 20} = (0.08)(0) + C$

=> $C = \ln\left(\frac{20}{K-20}\right)$

=> $\ln\frac{P}{K-P} = 0.08t + \ln\left(\frac{20}{K-20}\right)$ (Subbing C back into integral)

=> $\ln\left(\frac{P}{K-P}\right) - \ln\left(\frac{20}{K-20}\right) = 0.08t$

=> $\ln\frac{P(K-20)}{20(K-P)} = 0.08t$

=> $\frac{P(K-20)}{20(K-P)} = e^{0.08t}$

=> $PK - 20P = (20K - 20P)e^{0.08t}$

=> $PK - 20P = 20Ke^{0.08t} - 20Pe^{0.08t}$

=> $PK - 20P + 20Pe^{0.08t} = 20Ke^{0.08t}$

=> $P(K - 20 + 20e^{0.08t}) = 20Ke^{0.08t}$

=> $P = \frac{20Ke^{0.08t}}{K - 20 + 20e^{0.08t}}$