1) Indices:

a) The Laws of Indices:

See Tables pg 21

1) $a^p \times a^q = a^{p+q}$ e.g. $4^4 \times 4^3 = 4^7$

2) $\frac{a^p}{a^q} = a^{p-q}$ $e.g\ \frac{5^3}{5^2} = 5^{3-2} = 5^1$

3) $(a^p)^q = a^{pq}$ $e.g\ (5^2)^3 = 5^6$

4) $a^0 = 1$ $e.g.\ 7^0 = 1$ or $(0.5)^0 = 1$

5) $a^{-p} = \frac{1}{a^p}$ $e.g.\ 3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

6) $(ab)^p = a^p b^p$ $e.g.\ (3x)^2 = 3^2x^2 = 9x^2$

7) $(\frac{a}{b})^p = \frac{a^p}{b^p}$ $e.g.\ (\frac{2}{3})^3 = \frac{2^3}{3^3} = \frac{8}{27}$

8) $a^{\frac{1}{2}} = \sqrt{a}$ $e.g.\ 9^{\frac{1}{2}} = \sqrt{9} = 3$

9) $a^{\frac{1}{3}} = \sqrt[3]{a}$ $e.g.\ 27^{\frac{1}{3}} = \sqrt[3]{27} = 3$

c) Table of the most Powers:

x	x¹	x²	x³	x⁴	x⁵	x⁶	x⁷	x⁸
2	2	4	8	16	32	64	128	256
3	3	9	27	81	243			
4	4	16	64	256				
5	5	25	125	625				
6	6	36	216					
7	7	49	343					
8	8	64	512					
9	9	81	729					
10	10	100	1000					

b) Solving equations with indices:

Steps:

1. Try and spot which powers you're dealing with, using the table below e.g. if you see a 9 and a 27 in the question, it would be powers of 3
2. Tidy up both sides of the equation into a single power using the laws of indices above. e.g. $5^x = 5^y$
3. If the bases are the same on both sides, you can now let the powers be equal to each other. i.e. x = y
4. Solve the simple equation to find your solution.

Example 1: Solve $3^x = 27\sqrt{3}$.

$3^x = 3^3 . 3^{\frac{1}{2}}$ ......using Law 8 above on the $\sqrt{3}$

$3^x = 3^{3 + {}^1/_2}$.........using Law 1

$3^x = 3^{7/2}$ ....tidying up the power into a single fraction

=> $x = {}^7/_2$ ....as the bases are equal

Example 2: Solve the equation

$$2^{2x+1} - 5(2^x) + 2 = 0$$

- Start by breaking up the power of the first term, using the Laws of Indices:
  - => $2^{2x}2^1 - 5(2^x) + 2 = 0$ (Using Law 1)
  - => $(2^x)^2 2^1 - 5(2^x) + 2 = 0$ (Using Law 3)
  - => $2(2^x)^2 - 5(2^x) + 2 = 0$ (Moving the 2^1 out in front)
- Now let $y = 2^x$:
  - => $2(y)^2 - 5(y) + 2 = 0$
  - => $2y^2 - 5y + 2 = 0$
  - => $(2y - 1)(y - 2) = 0$
  - => $y = \frac{1}{2}$ or $y = 2$
- Now go back to our substitution to solve for the two x values:

If $y = \frac{1}{2}$	If $y = 2$
=> $2^x = \frac{1}{2^1}$	=> $2^x = 2^1$
=> $2^x = 2^{-1}$	=> $x = 1$
=> $x = -1$	

2) Logarithms:

a) Definition of Logs:

Notes:

- **Logarithms** are the opposite of raising a number to a particular power.
- The definition of logs is:

$$a^m = n \Leftrightarrow \log_a n = m$$

Example: If $2^5 = 32$, then we can write that using logs by writing: $\log_2 32 = 5$.

b) The Laws of Logs:

1) $\log_a(xy) = \log_a x + \log_a y$ e.g. $\log_2(20) = \log_2 5 + \log_2 4$

2) $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$ $e.g\ \log_5\left(\frac{2}{3}\right) = \log_5 2 - \log_5 3$

3) $\log_a(x^p) = p\log_a x$ $e.g\ \log_2 x^3 = 3\log_2 x$

4) $\log_a 1 = 0$ $e.g.\ \log_4 1 = 0$

5) $\log_a\left(\frac{1}{x}\right) = -\log_a x$ $e.g.\ \log_2\left(\frac{1}{8}\right) = -\log_2 8$

6) $\log_a(a^x) = x$ $e.g.\ \log_4(4^x) = x$

7) $a^{\log_a x} = x$ $e.g.\ 2^{\log_2 x} = x$

8) $\log_b x = \frac{\log_a x}{\log_a b}$ $e.g. \log_8 x = \frac{\log_2 x}{\log_2 8}$

See Tables pg 21

c) Simplifying Expressions:

Example: Simplify $\log_3 2 + 2\log_3 3 - \log_3 18$.

$\log_3 2 + 2\log_3 3 - \log_3 18$

$= \log_3 2 + 2(1) - [\log_3 9 + \log_3 2]$ (Laws 1, 6)

$= \log_3 2 + 2 - \log_3 9 - \log_3 2$

$= 2 - \log_3 9$ (as $\log_3 2 - \log_3 2 = 0$)

$= 2 - 2$ (as $\log_3 9 = 2$)

$= 0$

d) Solving Log Equations:

Notes:

- When solving equations with logs, use:

  If $\log_a b = \log_a c$
  => b = c.

- Solutions of log equations always have to be checked.

Example: Solve the equation $\log_3(x+1) - \log_3(x-1) = 1$.

- Aim is to tidy up both sides into a single log, and use rule above.
- Use Law 2 to tidy up the LHS, and Law 6 to rewrite '1' on the RHS:

$\log_3(x+1) - \log_3(x-1) = 1$

=> $\log_3 \frac{x+1}{x-1} = \log_3 3$

=> $\frac{x+1}{x-1} = 3$

=> $x + 1 = 3x - 3$

=> $2x = 4$

=> $x = 2$

Check:

$\log_3(x+1) - \log_3(x-1) = 1$

=> $\log_3(2+1) - \log_3(2-1) = 1$

=> $\log_3(3) - \log_3(1) = 1$

=> $1 - 0 = 1$

=> $1 = 1$

e) Equations involving Change of Base Law:

Notes:

- Easier to change higher bases to lower bases

Example: Solve the equation

$\log_2(x+1) + \log_4(x) = \log_4(x^3+1)$.

- Change all logs to base 2 first:

=> $\log_2(x+1) + \frac{\log_2(x)}{\log_2 4} = \frac{\log_2(x^3+1)}{\log_2 4}$

=> $\log_2(x+1) + \frac{\log_2(x)}{2} = \frac{\log_2(x^3+1)}{2}$

=> $2\log_2(x+1) + \log_2(x) = \log_2(x^3 + 1)$

- RHS is a single log as it is, but we have a few more steps to do to the LHS to tidy it up into a single log:

=> $\log_2(x+1)^2 + \log_2(x) = \log_2(x^3 + 1)$

=> $\log_2(x+1)^2(x) = \log_2(x^3 + 1)$

- Can now eliminate the logs, as we have a single log on both sides, with the same power:

=> $(x)(x+1)^2 = x^3 + 1$

=> $(x)(x^2 + 2x + 1) = x^3 + 1$

=> $x^3 + 2x^2 + x = x^3 + 1$

=> $2x^2 + x - 1 = 0$

=> $x = \frac{1}{2}$ $or\ x = -1$

- Checking the answers eliminates $x = \frac{1}{2}$ as a solution, so the answer is x = -1

e) Problem Solving with Logs:

Example 1: Population, P, is modelled as $P = 12300(e^{0.073t})$, where t is in years. After how many years, will the population reach 20,000 people?

$P = 12300(e^{0.073t})$

=> $20000 = 12300(e^{0.073t})$

=> $e^{0.073t} = \frac{20000}{12300} = 1.626$ (dividing both sides by 12300)

=> $\ln e^{0.073t} = \ln 1.626$ (taking natural log of both sides to eliminate e)

=> $0.073t = \ln 1.626$ (using Law 6 as $\ln e = 1$)

=> $t = \frac{\ln 1.626}{0.073} = \frac{0.486}{0.073} = 6.7yrs$

Example 2: Amount of radioactive tracer remaining after t days is given by $A = A_o(e^{-0.058t})$. A_o = starting amount. How many days, will it take for one half of the original amount to decay?

- Key in this type of question is we want **one half of the original amount**

=> $A = \frac{A_o}{2}$ or $A_o = 2A$

$A = A_o(e^{-0.058t})$

$A = 2A(e^{-0.058t})$ (filling in $A_o = 2A$)

=> $1 = 2(e^{-0.058t})$ (dividing both sides by A)

=> $0.5 = e^{-0.058t}$ (dividing both sides by 2)

=> $\ln 0.5 = \ln e^{-0.058t}$ (taking ln of both sides)

=> $\ln 0.5 = -0.058t$ (using Law 6 as $\ln e = 1$)

=> $t = \frac{\ln 0.5}{-0.058} = 11.95 = 12days$ (dividing both sides by -0.058)

15) Surds:

a) Surds:

Notes:

- A **surd** is a number in the form $\sqrt{\ }$ that **can't be written** as a **rational** number i.e. in the form $\frac{a}{b}$
  E.g. $\sqrt{2}$ and $\sqrt{3}$ are both surds but $\sqrt{9}$ is not as it can be written as $\frac{3}{1}$
- We can add/subtract similar surds together
  E.g. i) $3\sqrt{2} + 5\sqrt{2} = 8\sqrt{2}$
  ii) $4\sqrt{3} + 2\sqrt{2}$ ....we can't add these together as the $\sqrt{\ }$ parts are different

b) Reducing Surds:

- We can use the rule $\sqrt{ab} = \sqrt{a}\sqrt{b}$ to reduce larger surds into a simpler form:

Example: Simplify $\sqrt{50} + \sqrt{32}$

- We use 50 = 25 x 2 rather than 10 x 5 as 25 is a square number)

$\sqrt{50} + \sqrt{32}$
$= \sqrt{25}\sqrt{2} + \sqrt{16}\sqrt{2}$
$= 5\sqrt{2} + 4\sqrt{2}$
$= 9\sqrt{2}$

c) Solving Surd Equations:

Notes:

- A general rule of thumb for solving equations with surds is to **square both sides**.
- When squared both sides, there is sometimes an incorrect answer introduced so we always have to check our answers when solving surd equations.

Example 1: Solve the equation $\sqrt{2x+3} = 3$.

$(\sqrt{2x+3})^2 = (3)^2$ (Square both sides)
$2x + 3 = 9$
=> $x = 3$

Check:
$\sqrt{2x+3} = 3$
$\sqrt{2(3)+3} = 3$
$\sqrt{9} = 3$

Example 2: Solve the equation $\sqrt{x+7} + \sqrt{x+2} = 5$.

- More than one surd => rearrange the equation so there is one surd on each side and then square both sides as before:

$\sqrt{x+7} = 5 - \sqrt{x+2}$
$(\sqrt{x+7})^2 = (5 - \sqrt{x+2})^2$
$x + 7 = 25 - 10\sqrt{x+2} + x + 2$
$x + 7 = x + 27 - 10\sqrt{x+2}$

- Bring surd to one side and everything else to the other side:

$10\sqrt{x+2} = 20$

- Square both sides again as before:

$(10\sqrt{x+2})^2 = (20)^2$
$100(x+2) = 400$
$100x = 200$
$x = 2$

Check:
$\sqrt{x+7} + \sqrt{x+2} = 5$
$\sqrt{(2)+7} + \sqrt{(2)+2} = 5$
$\sqrt{9} + \sqrt{4} = 5$
$3 + 2 = 5$
$5 = 5$