Topic 6: Calculus

1) The Basics:

a) Differentiating Expressions:

Notes:

- Symbols: $\frac{dy}{dx}$ or $\mathbf{f}'(x)$
- The derivative of a constant = 0
- In general, to differentiate, we:

Multiply by the power and reduce the power by 1.

b) Second Derivative: (Differentiating twice)

Examples:

i) $y = 3x^3 - 4x^2 + 5x$ => $\frac{dy}{dx} = 27x^2 - 8x + 5$ => $\frac{d^2y}{dx^2} = 54x - 8$	ii) $f(x) = 2x^4 + 3x^2 - 3x + 2$ => $f'(x) = 8x^3 + 6x - 3$ => $f''(x) = 24x^2 + 6$

2) Slopes of Tangents:

Slope of the Tangent to the Curve = $\frac{dy}{dx}$

Example 1: If $y = 3x^2 + 5x - 8$, find the slope of the tangent to the curve at the point (-1, 2).

$y = 3x^2 + 5x - 8$

=> $\frac{dy}{dx} = 6x + 5$

@ (-1, 2)

=> $\frac{dy}{dx} = 6(-1) + 5 = -1$

Example 2: If $y = 2x^3 - 4x + 1$, find the equation of the tangent to the curve at the point (-2, -6).

- Find slope first: $\frac{dy}{dx} = 6x^2 - 4$
- Find slope at the point (-2, -6): $\frac{dy}{dx} = 6(-2)^2 - 4 = 20$
- Find the equation using equation formula and the point (-2,-6):

$$y - y_1 = m(x - x_1)$$
$$y - (-6) = 20(x - (-2))$$
$$y + 6 = 20(x + 2)$$
$$y + 6 = 20x + 40$$
$$y = 20x + 34$$

3) Curve Sketching:

a) Increasing/Decreasing Functions:

Increasing => $\frac{dy}{dx} > 0$

Decreasing => $\frac{dy}{dx} < 0$

Example: Find the range of values of x for which the curve $f(x) = x^2 - 3x + 4$ is increasing.

$f'(x) = 2x - 3$

Increasing => $f'(x) > 0$

$2x - 3 > 0$

$2x > 3$

=> $x > \frac{3}{2}$

b) Max/Min Points (Turning Points):

Max/Min Points => $\frac{dy}{dx} = 0$

Example: Find the max/min points of the curve $f(x) = x^3 - 2x^2 + 4$.

$f'(x) = 3x^2 - 4x$

Max/Min => $f'(x) = 0$

$3x^2 - 4x = 0$

$x(3x - 4) = 0$

$x = 0 \quad or\ x = \frac{4}{3}$

Sub into function at start to find y values of 4 and $\frac{76}{27}$.

=> Turning Points are $(0,4)$ and $(\frac{4}{3}, \frac{76}{27})$

4) Rates of Change:

a) Max/Min Problems:

Steps:

1. Get an expression for quantity to be maximised/minimised.
2. Differentiate.
3. Let derivative = 0 and solve to find max/min value.
4. Sub max/min value back into expression from step 1, if needed.

Example: A farmer want to enclose a field with 100m of fencing. Find the maximum area of the field.

Let Width = x => Length = 50 - x

=> Area = L x W = x(50 - x) = $50x - x^2$

=> A = $50x - x^2$

=> $\frac{dA}{dx} = 50 - 2x$ (differentiating expression for area)

50 - 2x = 0

=> x = 25

=> Max Area will be $50(25) - (25)^2 = 625$m^2

b) Rates of Change:

Tip:

Differentiate the expression for distance to get expressions for speed and acceleration.

Distance → Speed → Acceleration

Example: A body moves a distance given by the function $s = t^3 - 3t^2 + 7$, find the body's acceleration after 3 seconds.

Distance = $t^3 - 3t^2 + 7$

=> Speed = $3t^2 - 6t$ (differentiating distance expression)

=> Acceleration = $6t - 6$ (differentiating speed expression)

So after 2 secs: Acceleration = $6t - 6$ = 6(3) - 6 = 12m/s^2.