Topic 3: Patterns/Sequences

1) Arithmetic Sequences/Series:

a) Linear(Arithmetic) Sequences:

Notes:

- A list of numbers where the **difference** between **each term** is the **same** every time.
  E.g. 3, 8, 13, 18, ………
- In Senior Cycle, we refer to these sequences as **Arithmetic** Sequences.
- The General Term for an Arithmetic sequence is:

$$T_n = a + (n - 1)d$$

See Tables pg 22

where 'a' is the first term and 'd' is the common difference between the terms.

Example: i) Find the General Term for the sequence 3, 8, 13, 18…..
ii) Find the 50th term, T_{50}.

a = 3 and d = 5

i) => $T_n = 3 + (n-1)5$ => $T_n = 3 + 5n - 5$ => $T_n = 5n - 2$	ii) $T_n = 5n - 2$ => $T_{50} = 5(50) - 2$ => $T_{50} = 250 - 2 = 248$

b) Arithmetic Series:

Notes:

- If we add the terms of an arithmetic sequence together, then we get an arithmetic **series**.
- We need to be able to find the sum of the first n terms of such a series, which we can find using:

$$S_n = \frac{n}{2}\{2a + (n-1)d\}$$

See Tables pg 22

where 'a' is the first term and 'd' is the common difference between the terms of the series.

Example: Find the sum of the first 20 terms of the series 2 + 6 + 10 + 14 + …….

a = 2 and d = 4

=> $S_{20} = \frac{20}{2}\{2(2) + (20-1)4\}$

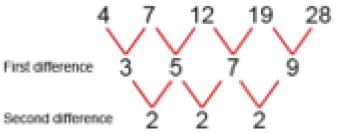
=> $S_n = 10\{4 + (19)4\}$

=> $S_n = 10\{80\} = 800$

2) Quadratic Sequences:

Notes:

- A sequence where the **second difference** is the **same** every time. E.g. 4, 7, 12, 19, 28……. (see below)



Steps to find General Term:

1. Let General Term = $T_n = an^2 + bn + c$
2. Find 2nd difference and let = 2a….solve for a.
3. Use any 2 terms to form two equations in b and c.
4. Solve both equations to find b and c.

Example: Find the General Term of the sequence 4, 7, 12, 19, 28

Step 1: Let the General Term $T_n = an^2 + bn + c$.

Step 2: Second difference = 2a = +2 => a = +1.

Step 3: Use two of the terms in the sequence to make two simultaneous equations, which we solve to find 'b' and 'c'……

$T_n = an^2 + bn + c$

$T_2 = (2)^2 + b(2) + c = 7$ => 4 + 2b + c = 7 => 2b + c = 3…..Eqn 1	$T_3 = (3)^2 + b(3) + c = 12$ => 9 + 3b + c = 12 => 3b + c = 3……Eqn 2

Step 4: Solving Equations 1 and 2 gives b = 0 and c = 3

=> $T_n = n^2 + (0)n + 3$

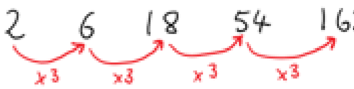
=> $T_n = n^2 + 3$

3) Geometric Sequences/Series:

a) Geometric Sequences:

Notes:

- A sequence where each term is found by **multiplying** the previous term by the same number every time.



- The General Term for a Geometric sequence is:

$$T_n = a.r^{n-1}$$

See Tables pg 22

where 'a' is the first term and 'd' is the common difference between the terms.

Example: i) Find the General Term for the sequence 3, 8, 13, 18…..
ii) Find the 50th term, T_{50}.

a = 3 and d = 5

i) => $T_n = 3 + (n-1)5$ => $T_n = 3 + 5n - 5$ => $T_n = 5n - 2$	ii) $T_n = 5n - 2$ => $T_{50} = 5(50) - 2$ => $T_{50} = 250 - 2 = 248$

b) Geometric Series:

Notes:

- If we add the terms of an geometric sequence together, then we get a geometric **series**.
- We need to be able to find the sum of the first n terms of such a series, which we can find using:

$$S_n = \frac{a(1-r^n)}{1-r} \text{ or } S_n = \frac{a(r^n - 1)}{r - 1} \text{ if } r > 1$$

See Tables pg 22

where 'a' is the first term and 'r' is the common ratio

Example: A geometric sequence is 3, 12, 48…….
i) Find the 10th term. ii) the sum of the first 10 terms.

i) Firstly, we will find the General Term:

- As a = 3 and r = 4, then

$T_n = ar^{n-1}$ => $T_n = 3(4^{n-1})$

- We can now find the 10th term by filling in n = 10:

$T_{10} = 3(4^{10-1}) = 3(4^9) = 786432$

ii) First, we need to find S_n:

$$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{3(4^n - 1)}{4 - 1} = \frac{3(4^n - 1)}{3} = 4^n - 1$$

=> $S_{10} = 4^{10} - 1 = 1{,}048{,}575$

4) Limits of a Sequence:

Notes:

- Sometimes a sequence can be approaching a particular number e.g. 1, $\frac{1}{2}$, $\frac{1}{4}$…… is a sequence that approaches 0.
- If a sequence approaches a certain number L, as the number of terms increases, then we say the sequence is **convergent**.
- Written mathematically as:

$$\lim_{n\to\infty} T_n = L$$

- If the sequence doesn't approach any value, then the sequence is said to be **divergent**.
- Another very useful property of limits is:

$$\lim_{n\to\infty} \frac{1}{n^p} = 0$$

Example: Evaluate $\lim_{n\to\infty} \frac{n^2}{3n^2+n}$.

- To evaluate the limit, we begin by dividing all terms by the highest power of n…….in this case n^2:

$$\lim_{n\to\infty} \frac{n^2}{3n^2 + n} = \lim_{n\to\infty} \frac{\frac{n^2}{\boldsymbol{n^2}}}{\frac{3n^2}{\boldsymbol{n^2}} + \frac{n}{\boldsymbol{n^2}}}$$

$$= \lim_{n\to\infty} \frac{1}{3 + \frac{1}{n}}$$

- We can now use the property above to evaluate the limit:

$$= \frac{1}{3+0} = \frac{1}{3}$$

5) Infinite Series/Cubic Sequences:

a) Infinite Series:

Notes:

- Series where the terms of an infinite sequence are added up.
- Only infinite Geometric Series are on the course e.g. 3 + 9 + 27 +…

$$S_\infty = \frac{a}{1-r}$$

See Tables pg 22

(for an infinite Geometric Series, where $|r| < 1$)

Example: Sum to infinity of the series $5 + \frac{5}{6} + \frac{5}{36} + \cdots$?

- Geometric Series with a = 5 and r = $\frac{1}{6}$

$$S_\infty = \frac{a}{1-r} = \frac{5}{1-\frac{1}{6}}$$

$$= \frac{5}{\frac{5}{6}}$$

$$= 6$$

Example 2: Express 2.7777 in the form $\frac{a}{b}$.

- Decimal in the form of a series:

2.777 = 2 + 0.7 + 0.07 + 0.007 + …

- Now isolate the part of the series after the '2':

2.777 = 2 + (0.7 + 0.07 + 0.007 + …)

- Terms in brackets are an infinite geometric series with a = 0.7 and r = $\frac{1}{10}$, so:

$$\frac{a}{1-r} = \frac{0.7}{1-0.1} = \frac{0.7}{0.9} = \frac{7}{9}$$

So, 2.777….. is: $2 + \frac{7}{9} = \frac{25}{9}$

b) Cubic Sequences:

Notes:

- A sequence where the **third difference** is the **same** every time.

E.g. 4, 14, 40, 88, 164……. (see below)