Formal Proofs for LCHL:

a) Proof by contradiction that $\sqrt{2}$ is irrational:

Assume $\sqrt{2}$ is not irrational => $\sqrt{2}$ can be written in the form $\frac{a}{b}$

=> $\sqrt{2} = \frac{a}{b}$ (where a and b have no common factor)

=> $2 = \frac{a^2}{b^2}$

=> $2b^2 = a^2$

As $2b^2$ is an even number => a^2 must be even

=> 'a' can be written in the form 2k

=> $2b^2 = (2k)^2$

=> $2b^2 = 4k^2$

=> $b^2 = 2k^2$, which means b is even as well.

If 'a' and 'b' are even, then 2 must divide into both => Contradiction

=> $\sqrt{2}$ is irrational

b) Example of Proof by Contradiction in Geometry:

- Assume opposite is true and prove that the opposite is impossible.

To prove: An equilateral triangle is also an acute-angled triangle (i.e. has no angle bigger than 90°)

Proof:

- Assume the opposite is true i.e. equilateral triangle is NOT right-angled
  => it has one angle bigger than 90°
- As triangle is equilateral, al angles are equal
  => it has 3 angles bigger than 90°

But……

Sum of 3 angles in a triangle is 180°, so this is impossible.

c) Derivation of Sum of Infinite Geometric Series Formula:

- Consider the Geometric Series a + ar + ar^2 + ar^3 + ……….
- The sum of the first term, sum of the first two terms, sum of the first three terms etc.

$S_1 = a$

$S_2 = a + ar$

$S_3 = a + ar + ar^2$

.

.

$S_n = a + ar + ar^2 + \ldots\ldots + ar^{n-1}$

- We can now find the limit of this series as n approaches infinity:

$$\lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$$

- As the $\frac{a}{1-r}$ is a constant, we can simply move that out in front of the limit:

$$\lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} \lim_{n\to\infty} (1-r^n)$$

- As long as r is between -1 and 1 i.e. $|r| < 1$, then the limit of r^n as n approaches infinity will be 0.
- So, the limit simplifies to:

$$\frac{a}{1-r}(1-0) = \frac{a}{1-r}$$

d) Proof of DeMoivre's Theorem by Induction:

To Prove: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

Step 1: Prove proposition is true for smallest value of n:

$(\cos\theta + i\sin\theta)^1 = \cos 1(\theta) + i\sin 1(\theta)$, which is true.

Step 2: Assume proposition is true for n = k:

$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

Step 3: Show proposition is true for n = k + 1:

$(\cos\theta + i\sin\theta)^{k+1} = \cos(k+1)\theta + i\sin(k+1)\theta$

LHS:

$(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k.(\cos\theta + i\sin\theta)^1$

$= (\cos k\theta + i\sin k\theta).(\cos\theta + i\sin\theta)$ (using P(k))

$= \cos k\theta\cos\theta + \cos k\theta i\sin\theta + i\sin k\theta\cos\theta + i^2\sin\theta\sin k\theta$ (multiplying out)

$= (\cos k\theta\cos\theta - \sin\theta\sin k\theta) + i[\cos k\theta\sin\theta + \cos\theta\sin k\theta]$ (put real and imag parts together)

$= (\cos(k\theta+\theta) + i\sin(k\theta+\theta))$ (as cos(A+B)=cosAcosB-sinAsinB and sin(A+B)=cosAsinB+sinAcosB)

$= (\cos(k+1)\theta + i\sin(k+1)\theta)$, which is = RHS above.

=> $P(k+1)\ is\ true, if\ P(k) is\ true$

=> $P(n) is\ true\ \forall\ n \in N$. Q.E.D.

e) Derivation of Amortisation Formula:

Proof:

- Let P = the amount borrowed, A = the repayment amount, t = the time period of repayment and i = the interest rate.
- The amount borrowed has to be equal to the present value of all the repayments:

$$=> P = \frac{A}{(1+i)^1} + \frac{A}{(1+i)^2} + \frac{A}{(1+i)^3} + \cdots + \frac{A}{(1+i)^t}$$

- A Geometric Series with a = $\frac{A}{(1+i)}$ and r = $\frac{1}{(1+i)}$, so:

$$=> P = \frac{\frac{A}{(1+i)}\left(1-\left(\frac{1}{(1+i)}\right)^t\right)}{1-\frac{1}{(1+i)}} \qquad \text{(Using Sn formula } \frac{a(1-r^n)}{1-r}\text{)}$$

$$=> P = \frac{\left(\frac{A}{(1+i)}\right)\left(1-\frac{1^t}{(1+i)^t}\right)}{1-\frac{1}{(1+i)}}$$

$$=> P = \frac{\left(\frac{A}{(1+i)}\right)\left(1-\frac{1^t}{(1+i)^t}\right)}{\frac{1(1+i)-1}{(1+i)}}$$

$$=> P = \frac{\left(\frac{A}{(1+i)}\right)\left(\frac{1(1+i)^t-1}{(1+i)^t}\right)}{\frac{i}{(1+i)}}$$

$$=> P = \left(\frac{A}{(1+i)}\right)\left(\frac{(1+i)^t-1}{(1+i)^t}\right) \times \frac{(1+i)}{i}$$

$$=> P = \left(\frac{A}{\cancel{(1+i)}}\right)\left(\frac{(1+i)^t-1}{(1+i)^t}\right) \times \frac{\cancel{(1+i)}}{i}$$

$$=> P = (A)\left(\frac{(1+i)^t-1}{i(1+i)^t}\right)$$

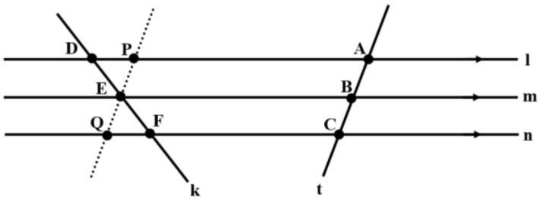
$$=> A = \frac{P(i(1+i)^t)}{(1+i)^t-1}$$

Q.E.D.

f) Theorem 11:

Theorem 11: If three parallel lines cut off equal segments on some transversal line, then they will cut off equal segments on any other transversal.

Diagram:



Given: Three parallel lines l, m and n, intersecting the transversal t at the points A, B and C such that |AB| = |BC|. Another transversal k intersects the lines at D, E and F.

To Prove: |DE| = |EF|

Construction: Through E, construct a line parallel to t and intersecting l at the point P and n at the point Q.

Proof: PEBA and EQCB are parallelograms (from construction)

=> |PE| = |AB| and |EQ| = |BC| (opposite sides of a parallelogram)

But |AB| = |BC| (Given)

So => |PE| = |EQ|.

Consider now △DEP and △FEQ:

|PE| = |EQ| (from above)

|∠PED| = |∠FEQ| (Vertically opposite angles)

|∠DPE| = |∠FQE| (Alternate angles)

=> △DEP and △FEQ are congruent (ASA)

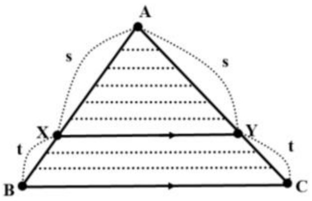
=> |DE| = |EF|.

Q.E.D.

g) Theorem 12:

Theorem 12: Let ABC be a triangle. If a line XY is parallel to BC and cuts [AB] in the ratio s : t, then it also cuts [AC] in the same ratio.

Diagram:



Given: The triangle ABC with XY parallel to BC.

To Prove: $\frac{|AX|}{|XB|} = \frac{|AY|}{|YC|}$

Construction: Divide [AX] into s equal parts and [XB] into t equal parts. Draw a line parallel to BC through each point of division.

The parallel lines make intercepts of equal length along the line [AC] (From Theorem 11)

=> [AY] is divided into s equal intercepts and [YC] is divided into t equal intercepts.

=> $\frac{|AY|}{|YC|} = \frac{s}{t}$

But $\frac{|AX|}{|XB|} = \frac{s}{t}$

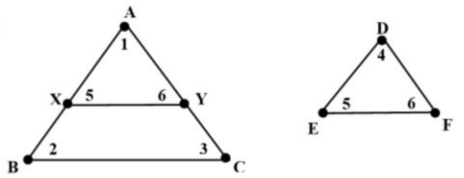
=> $\frac{|AX|}{|XB|} = \frac{|AY|}{|YC|}$

Q.E.D.

h) Theorem 13:

Theorem 13: If two triangles ABC and DEF are similar, then their sides are proportional in order.

Diagram:



Given: Similar triangles ABC and DEF

To prove: $\frac{|AB|}{|DE|} = \frac{|BC|}{|EF|} = \frac{|AC|}{|DF|}$

Construction: Mark the point X on [AB] such that |AX| = |DE|.
Mark the point Y on [AC] such that |AY| =|DF|.
Join XY.

Proof: △AXY and △DEF are congruent. (SAS)

=> |∠AXY| = |∠DEF| = |∠5| (Corresponding angles)

=> |∠AXY| = |∠ABC| (as triangles ABC and DEF are similar)

=> XY || BC (as angles 2 and 5 are corresponding)

=> $\frac{|AB|}{|AX|} = \frac{|AC|}{|AY|}$ (A line parallel to one side divides other side in the same ratio….Theorem 12)

=> $\frac{|AB|}{|DE|} = \frac{|AC|}{|DF|}$

Similarly, it can be proven that $\frac{|AB|}{|DE|} = \frac{|BC|}{|EF|}$

=> $\frac{|AB|}{|DE|} = \frac{|BC|}{|EF|} = \frac{|AC|}{|DF|}$

Q.E.D.

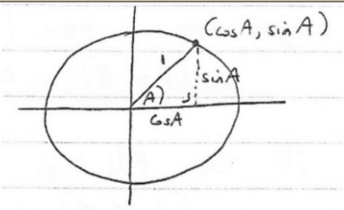
i) Proof of Trig Identities:

1) To prove: $\cos^2 A + \sin^2 A = 1$

Using Pythagoras Theorem:

$$(\cos A)^2 + (\sin A)^2 = (1)^2$$

=> $\cos^2 A + \sin^2 A = 1$



2) To prove: $\cos(A - B) = \cos A \cos B + \sin A \sin B$

Let $P(\cos A, \sin A)$ and $Q(\cos B, \sin B)$ be two points on a unit circle.

Using distance formula:

$$|PQ| = \sqrt{(\cos A - \cos B)^2 + (\sin A - \sin B)^2}$$

$$|PQ|^2 = \cos^2 A + \cos^2 B - 2\cos A \cos B + \sin^2 A + \sin^2 B - 2\sin A \sin B$$

$$= (\cos^2 A + \sin^2 A) + (\cos^2 B + \sin^2 B) - 2(\cos A \cos B + \sin A \sin B)$$

$$= 1 + 1 - 2(\cos A \cos B + \sin A \sin B)$$

$$= 2 - 2(\cos A \cos B + \sin A \sin B)$$

Using Cosine Rule to find |PQ| instead:

$$|PQ|^2 = (1)^2 + (1)^2 - 2(1)(1)\cos(A - B)$$

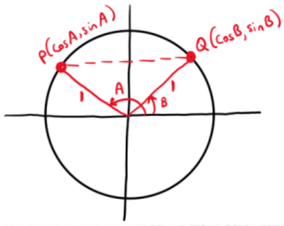
=> $|PQ|^2 = 2 - 2\cos(A - B)$

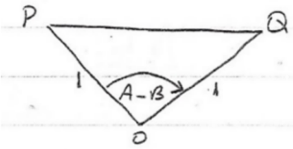
Equating the two expressions for $|PQ|^2$:

$$2 - 2\cos(A - B) = 2 - 2(\cos A \cos B + \sin A \sin B) \qquad (-2)$$

=> $-2\cos(A - B) = -2(\cos A \cos B + \sin A \sin B) \qquad (\div -2)$

=> $\cos(A - B) = \cos A \cos B + \sin A \sin B$





3) To prove: $\cos(A + B) = \cos A \cos B - \sin A \sin B$

We know from (2) that:

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

If we fill in $-B$ instead of B:

$$\cos(A - (-B)) = \cos A \cos(-B) + \sin A \sin(-B)$$

=> $\cos(A + B) = \cos A \cos(-B) - \sin A \sin(-B)$

Since $\cos(-B) = \cos(B)$ and $\sin(-B) = -\sin(B)$

=> $\cos(A + B) = \cos A \cos B + \sin A \sin B$

4) To prove: $\sin(A + B) = \sin A \cos B + \cos A \sin B$

We know from (2) that $\cos(A - B) = \cos A \cos B + \sin A \sin B$

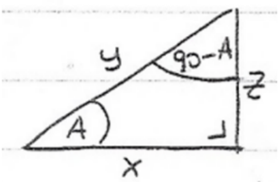
If we fill in $90 - A$ instead of A, we get:

$$\cos((90 - A) - B) = \cos(90 - A)\cos B + \sin(90 - A)\sin B$$

From the diagram on the right:

$$\sin(90 - A) = \frac{x}{y} = \cos A$$

$$\cos(90 - A) = \frac{z}{y} = \sin A$$



=> $\cos((90 - A) - B) = \sin A \cos B + \cos A \sin B$

=> $\cos(90 - (A + B)) = \sin A \cos B + \cos A \sin B$

=> $\sin(A + B) = \sin A \cos B + \cos A \sin B$

5) To prove: $\sin(A - B) = \sin A \cos B - \cos A \sin B$

We know from (4) that $\sin(A + B) = \sin A \cos B + \cos A \sin B$

If we fill in $-B$ instead of B:

$$\sin(A + (-B)) = \sin A \cos(-B) + \cos A \sin(-B)$$

=> $\sin(A - B) = \sin A \cos(-B) + \cos A \sin(-B)$

Since $\cos(-B) = \cos(B)$ and $\sin(-B) = -\sin(B)$

=> $\sin(A - B) = \sin A \cos B - \cos A \sin B$

6) To prove: $\cos 2A = \cos^2 A - \sin^2 A$

We know from (3) that $\cos(A + B) = \cos A \cos B - \sin A \sin B$

If we replace B by A we get:

$$\cos(A + A) = \cos A \cos A - \sin A \sin A$$

=> $\cos 2A = \cos^2 A - \sin^2 A$

7) To prove: $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$$\tan(A + B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B + \sin A \sin B}$$ Using (3) and (4)

We now divide each of the four terms by $\cos A \cos B$:

$$\tan(A + B) = \frac{\frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} + \frac{\sin A \sin B}{\cos A \cos B}}$$

=> $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

8) To prove: Sine Rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Area of the triangle shown = $\frac{1}{2} ab \sin C$

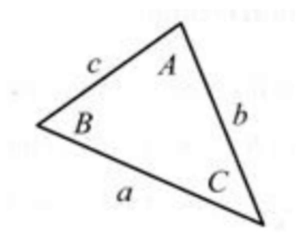
$$\frac{1}{2} ab \sin C = \frac{1}{2} ac \sin B = \frac{1}{2} bc \sin A$$

Dividing all three by $\frac{1}{2} abc$ gives:

$$\frac{\frac{1}{2} ab \sin C}{\frac{1}{2} abc} = \frac{\frac{1}{2} ac \sin B}{\frac{1}{2} abc} = \frac{\frac{1}{2} bc \sin A}{\frac{1}{2} abc}$$

=> $\frac{\sin C}{c} = \frac{\sin B}{b} = \frac{\sin A}{a}$

=> $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$



9) To prove: Cosine Rule: $a^2 = b^2 + c^2 - 2bc \cos A$

Consider the triangle shown on the right.

Using the distance formula to find a:

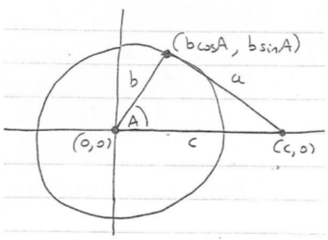
$$a = \sqrt{(b \cos A - c)^2 + (b \sin A - 0)^2}$$

=> $a^2 = b^2 \cos^2 A + c^2 - 2bc \cos A + b^2 \sin^2 A$

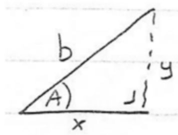
=> $a^2 = b^2(\cos^2 A + \sin^2 A) + c^2 - 2bc \cos A$

=> $a^2 = b^2(1) + c^2 - 2bc \cos A$

=> $a^2 = b^2 + c^2 - 2bc \cos A$



Note:



$\cos A = \frac{x}{b}$

=> $x = b \cos A$

$\sin A = \frac{y}{b}$

=> $y = b \sin A$