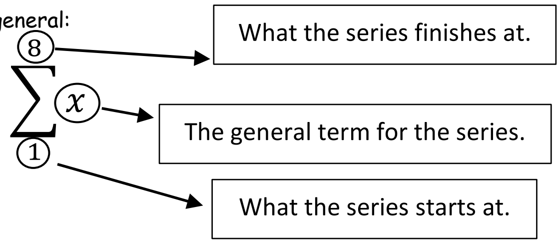
Topic 4: Proof by Induction:

a) Summations:

Notes:

- A shorter way of writing a series is to use symbol sigma $\sum$.
- In general:



Examples: i) Evaluate $\sum_1^6 n^2$.

The general term for this pattern is n^2 and it starts at n = 1 and ends at n = 6, so:

$$\sum_1^6 n^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = 91$$

b) Proof by Induction:

Steps:

1. Prove proposition is true for smallest value of n (usually n = 1)…P(1)
2. Assume proposition is true for n = k……P(k)
3. Show proposition is true for n = k + 1 i.e. P(k) => P(k + 1)

Notes:

- There are four different applications of Induction that we need to be familiar with:
  - a) Divisibility
  - b) Series
  - c) Inequalities
  - d) Proof of DeMoivre's Theorem

c) Divisibility:

Example: Prove that $P(n) = 13^n - 5^n$ is divisible by 8, $n \in N$.

Step 1: $P(1) = 13^1 - 5^1 = 8$, which is divisible by 8, so P(1) is true.

Step 2: Assume $P(k) = 13^k - 5^k$ is divisible by 8

Step 3: $P(k+1) = 13^{k+1} - 5^{k+1}$

- Now manipulate $P(k+1)$ to try and link it to $P(k)$:

$$13^{k+1} - 5^{k+1} = 13^k 13^1 - 5^k 5^1$$

- We now split up this 13 into (8 + 5):

$$13^k 13^1 - 5^k 5^1$$
$$= 13^k(8+5) - 5^k 5^1$$
$$= 13^k(8) + 13^k(5) - 5^k 5^1$$

- Factorise out the 5 out of the second and third terms:

$$= 13^k(8) + 5(13^k - 5^k)$$

- Expression in the second set of brackets is $P(k)$, which we have assumed already is true.
- 1st term is divisible by 8, as there is a multiplication by 8 in it.
- Both terms of $P(k+1)$ divisible by 8 => whole thing divisible by 8.
- To finish our proof, we write:

=> $P(k+1)$ *is true, if* $P(k)$ *is true*

=> $P(n)$ *is true* $\forall\, n \in N$. Q.E.D

d) Series:

Example: Prove that 1 + 3 + 5 + …… + (2n – 1) = n^2

Step 1: If n = 1, then that means the first term on the LHS, so:

1 = $(1)^2$ is true

Step 2: Assume $1 + 3 + 5 + \cdots + (2k-1) = k^2$

Step 3: First, fill in n = k + 1, so our series becomes:

1 + 3 + 5 + …… + (2k – 1) + (2(k + 1) – 1) = $(k + 1)^2$

=> 1 + 3 + 5 + …..+ (2k – 1) + (2k + 1) = $(k + 1)^2$

- But all the terms underlined above in red are P(k), which we know from the previous step, is equal to k^2, so:

k^2 + (2k + 1) = $(k + 1)^2$

- And now squaring out the right-hand side gives:

k^2 + 2k + 1 = k^2 + 2k + 1

- Again, we finish by stating:

=> $P(k+1)$ *is true, if* $P(k)$ *is true*

=> $P(n)$ *is true* $\forall\, n \in N$. Q.E.D.

e) Inequalities:

Example: Prove that $2n^2 > (n+1)^2$ for $n \geq 3$.

Step 1: Smallest n can be is 3, so:

$$2(3)^2 > (3+1)^2$$
$$18 > 16, \text{ which is true}$$

Step 2: Assume $2k^2 > (k+1)^2$

Step 3: First, we will fill in n = k + 1, so our series becomes:

$$2(k+1)^2 > (k+1+1)^2$$
$$\Rightarrow 2(k+1)^2 > (k+2)^2$$

- Let's multiply out both sides:

$$2(k^2 + 2k + 1) > k^2 + 4k + 4$$
$$\Rightarrow 2k^2 + 4k + 2 > k^2 + 4k + 4$$

- From P(k), we know that $2k^2 > (k+1)^2$, so we can replace the $2k^2$ with $(k+1)^2$ and the inequality should still hold:

$$(k+1)^2 + 4k + 2 > k^2 + 4k + 4$$
$$\Rightarrow k^2 + 2k + 1 + 4k + 2 > k^2 + 4k + 4$$

=> $2k + 1 + 4k + 2 > 4k + 4$ (taking k^2 from both sides)

=> $6k - 4k > 4 - 3$

=> $2k > 1$, which is always true for any k value greater than 3.

=> $P(k+1)$ *is true, if* $P(k)$ *is true*

=> $P(n)$ *is true* $\forall\, n \in N$. Q.E.D.