Topic 5: Financial Maths

1) The Basics:

a) Percentages:

To find the percentage of a number:

Example: Find 24% of 250.

Method 1: Calculate $\frac{24}{100} x \frac{250}{1} = 60$

Method 2: Multiply 250 by 0.24 = 60

To find the total when given percentage:

Example: 25% of the marks in an exam are going for the practical part. If there are 50 marks for the practical, how many marks is the whole exam worth?

Steps:

1) Let % = value you're given

25% = 50

2) Find what 1% represents by dividing both sides

1% = $\frac{50}{25} = 2$

3) Find 100% by multiplying by 100:

100% = 2 x 100 = 200marks

Note:

In this particular example, we could also have just multiplied 50 by 4, as 25% represents $^1/_4$ of the total marks

b) % Profit / Loss / Discount:

% Profit or Mark-Up = $\frac{Profit}{Cost\ Price} x\ 100\ \%$

% Profit Margin = $\frac{Profit}{Selling\ Price} x\ 100\ \%$

% Loss = $\frac{Loss}{Cost\ Price} x\ 100\ \%$

% Discount = $\frac{Discount}{Cost\ Price} x\ 100\ \%$

c) VAT:

VAT excluded:

Example: Bill comes to €120. Find final bill with 13.5% VAT.

VAT = 13.5% of 120 = 120 x 0.0135 = €16.20

=> Final Bill = €120 + €16.20 = €136.20

VAT included:

Example: Bill including VAT comes to €340.50. Find bill without VAT, if VAT is 13.5%.

Bill + VAT = €340.50

=> 113.5% = €340.50

=> 1% = €3

=> 100% = €300

d) Foreign Exchange/Unit Conversions:

Steps:

1. Write the conversion with the currency you want on the right.
2. Get a 1 on the left-hand side, by dividing both sides.
3. Multiply both sides to get the value you want.

Example: If €1 = $1.32, how many euro would you get for $200?

Step 1: Put euro on the right

$1.32 = €1

Step 2: Get a 1 on the left-hand side

$1 = € $\frac{1}{1.32}$ (dividing both sides by 1.32)

Step 3: Multiply both sides

$200 = $\frac{1}{1.32} x\ 200$ = €151.52

Note:

This method can be used for any units conversions e.g. miles to kilometres.

e) Household Bills:

Notes:

- For utilities (e.g.s. gas, electricity): charge per unit used
- To calculate the units used, subtract the meter readings
- Standing charge and VAT also has to be added on.
- With Gas Bills, there is also a Carbon Tax that needs to be added on.

Example: Cost of electricity with meter readings of 21310 and 21836. A standing charge of €21.60. Cost per unit is €0.15. VAT of 13.5%.

Units used = 21836 – 21310 = 526 units

Cost for electricity = 526 x €0.15 = €78.90

Standing Charge = €21.60

=> Total Before VAT = €78.90 + €21.60 = €100.50

VAT = 13.5% of €100.50 = €13.57

=> Final Bill = €100.50 + €13.57 = €114.07

2) Income Tax:

a) Income Tax Terminology:

- **Gross Income:** total pay someone gets before any taxes or deductions are taken
- **Net Income:** Take home pay or pay that we get after all taxes and deductions
- Rates Of Tax: **Higher** Rate (usually about 42%) and **Standard** Rate (usually about 20%)
- **Standard Rate Cut-Off Point:** Anything you earn up to this is taxed at the standard rate of tax
- **Gross Tax:** Total tax owing to the government before credits are deducted
- **Tax Credits:** Money deducted from the gross tax
- **Tax Payable:** Tax that you actually pay after credits have been subtracted
- **Statutory Deductions:** Payments that you have to make to the government e.g. income tax (P.A.Y.E.)
- **Non-statutory Deductions:** Voluntary deductions that somebody pays e.g.s trade union fees or health insurance

b) Answering Questions:

- The questions are nearly always made up of 3 parts:
  - **Part 1:** Calculation of Gross Tax by adding….
    Tax @ Lower Rate + Tax @ Higher Rate
  - **Part 2:** Calculation of Tax Payable using the equation
    Tax Paid = Gross Tax - Tax Credits
  - **Part 3:** Working out Net Income by taking off all deductions including Tax Paid, USC and PRSI (See below), Union Fees, Health Insurance etc.

c) USC/PRSI:

USC: Have to be given the rates and bands:

- 2% of the first €10036 = €200.72
- 4% of the next €5980 = €239.20
- 7% on the balance of income => need to subtract (10036 + 5980) from Gross Income and then get 7%

PRSI:

- Usually in class A, €127/week is free of PRSI deductions
  => €127 x 52 = $6604 (needs to be taken from gross income)
- Then pay 4% on the remainder of your income.

3) Compound Interest/Depreciation:

a) Terminology:

- **Principal:** Amount of money invested or borrowed
- **Interest:** Money added by the bank
- **Rate:** what percentage the interest is added at
- **Amount or Final Value:** The value of money at the end of the term it has been borrowed or invested for.

b) Compound Interest:

Notes:

- Allows us to compare sums of money in the future with their present value.
- The formula takes into account the **time value of money**.

Method 1: Used if rates change from year to year or payments/withdrawals are being made between years

- Lay out Year 1, Year 2, Year 3 etc.
- Work out interest each year and add to Principal at start of the year

Method 2: Formula

$$F = P(1+i)^t$$

See Tables pg 30

where **F** is the Amount, **P** is the Principal, **i** is the Rate of Interest as a **decimal** (e.g. 3% = 0.03) and **t** is the time in years the money is invested/borrowed for.

c) Depreciation: (items losing value)

- The formula on the right can be used for Depreciation problems…….just replace the '+' with a '-'……see below.
- The rate must be the same each year to use the formula.

$$F = P(1-i)^t$$

where **F** is the Final value, **P** is the starting value, **i** is the Rate of Depreciation as a **decimal** (e.g. 2.5% = 0.025) and **t** is the time in years.

4) Net Present Value/Bonds:

a) Net Present Value (NPV):

Notes:

- In order to calculate **Net Present Value (NPV)** we use:

Net Present Value = Present value of all cash inflows – Present value of all cash outflows

- If the **NPV** $>$ **0** => profit made => a **good investment**.
- If the **NPV** $<$ **0** => loss made => a **bad investment**.

Example: For an initial investment of €30,000 at the beginning of the year one a venture capitalist will get the pay-out detailed below. The discount rate for each year of the project is 5%. Should he/she accept the deal?

YEAR	CASH FLOW (at the end of each yr)
1	€10,000
2	€12,000
3	€15,000

- Firstly, we need to bring all future payments back to present values to compare them:

Payment 1: $P = \frac{F}{(1+i)^t} = \frac{10000}{(1+0.05)^1} = €9523.81$

Payment 2: $P = \frac{F}{(1+i)^t} = \frac{12000}{(1+0.05)^2} = €10884.35$

Payment 1: $P = \frac{F}{(1+i)^t} = \frac{15000}{(1+0.05)^3} = €12957.56$

- So, the present value of all the cashflows above is

$$= 9523.81 + 10884.35 + 12957.56$$
$$= €33365.72$$

- We can now calculate the NPV:

$$= €33365.72 - €30000 = €3365.72$$

- As the NPV is positive, the venture capitalist should invest.

b) Bonds

Notes:

- A cash payment made to the government or private company for an agreed number of years.
- Investor paid a fixed sum at the end of each year.
- Government or company also repays the original value of the bond to the investor with the final payment.

Example 1: An investment bond offers a 20% return at the end of 5 years. Calculate the AER for this bond.

- A 20% return means that the final value of the investment will be 120% of the original money invested.

$F = P(1+i)^t$

=> $1.2 = 1(1+i)^5$

=> $\sqrt[5]{1.2} = 1 + i$

=> $1.037137289 = 1 + i$

=> $i = 3.7\%$

Example 2: 8-year bond of €2500 pays €150 at the end of each year at an AER of 4%. What is the cost of the bond?

- Calculate the present value of €2500 in 8 years' time:

$F = P(1+i)^t$

=> $2500 = P(1+0.04)^8$

=> $P = \frac{2500}{(1+0.04)^8} = €1826.73$

- Now calculate the present value of the eight payments of €150:

$$\frac{150}{1.04} + \frac{150}{(1.04)^2} + \cdots + \frac{150}{(1.04)^8}$$

- Geometric Series with a = $\frac{150}{1.04}$ and r = $\frac{1}{1.04}$

=> $S_n = \dfrac{a(1-r^n)}{1-r}$

=> $S_8 = \dfrac{\frac{150}{1.04}\left(1-\left(\frac{1}{1.04}\right)^8\right)}{1-\frac{1}{1.04}}$

=> $S_8 = \dfrac{\frac{150}{1.04}\left(1-\left(\frac{1}{1.04}\right)^8\right)}{1-\frac{1}{1.04}} = €1009.91$

- We can now calculate the total cost of the bond:
  Total Cost = €1826.73 + €1009.91 = €2836.64

5) Applications of Geometric Series:

a) Converting AER to a monthly rate:

Notes:

- **Annual Equivalent Rate (AER)** tells you how much interest your money would earn in exactly one year, regardless of how long you invested it for.
- **Annual Percentage Rate (APR)** similar to the AER but takes set-up fees and other costs into account so customers can compare and see which loans are more expensive.
- We have two options when handling monthly calculations involving AERs:
  - i) converting to a monthly rate (See Example 1)
  - ii) convert the times to years by dividing them by 12 (See Example 2)

Example 1: An AER of 3.6% means that if we put in a sum of money at the start of the year, there would be 103.6% of that sum by the end of the year i.e. 100% (1.0) becomes 103.6% (1.036) over 12 months.

- To convert the AER to a monthly rate first.

$$F = P(1+i)^t$$

=> $1.036 = 1(1+i)^{12}$

=> $\sqrt[12]{1.036} = 1 + i$ (12th root of both sides)

=> $1.00295 = 1 + i$

=> $1.00295 - 1 = i$

=> $0.00295 = i$ (=> $i = 0.295\%$)

Example 2: Shauna needs €10,000 in five years' time. How much should she deposit at the end of each month in an account that pays 4% AER to achieve this?

- Use the compound interest formula, but we don't know what 'P' is:

$$F = P(1+i)^t$$

- First deposit made at the end of the 1st month so that deposit will be in the account for 59 months (5 years = 60 months).
- Next deposit will be in the account for 58 months etc. so:

=> $10000 = P(1.04)^{\frac{59}{12}} + P(1.04)^{\frac{57}{12}} + \cdots + P(1.04)^{\frac{1}{12}} + P$

- Factorise out the P, and rearrange terms in ascending order:

=> $10000 = P[1 + (1.04)^{\frac{1}{12}} + (1.04)^{\frac{2}{12}} + \cdots + (1.04)^{\frac{59}{12}}]$

- Brackets contains a Geometric Series with 60 terms that has a = 1 and r = $(1.04)^{\frac{1}{12}}$, so we can sum those using the formula:

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

=> $S_{60} = \frac{1[((1.04)^{\frac{1}{12}})^{60} - 1]}{((1.04)^{\frac{1}{12}} - 1)}$ => $S_{60} = \frac{1[0.2166529024]}{0.003273739} = 66.179$

- So, finally:

$10000 = P(66.179)$ => $P = \frac{10000}{66.179} = €151.11.$

b) Investments/Savings:

Notes:

Example 1: A credit union offers a savings account with an AER of 3.6%. A man deposits €175 at the end of each month, starting 31 January 2018. How much will the investment be worth on 31 December 2022?

- In this example, he will be making deposits of €175 for 60 months.
- The 1st €175 will be in his account for 59 months (as it's being deposited at the end of the month), the 2nd for 58 months etc. and the last deposit will go in on the last day of December 2022, so it won't be in the account for any length of time.
- So, the deposits over the 5 years can be represented by the series:

$$F = 175(1.00295)^{59} + 175(1.00295)^{58} + \cdots + 175(1.00295)^1 + 175$$

=> $F = 175[(1.00295)^{59} + (1.00295)^{58} + \cdots + (1.00295)^1 + 1]$

=> $F = 175[1 + (1.00295)^1 + \cdots + (1.00295)^{58} + (1.00295)^{59}]$

- And evaluating the sum of the Geometric Series inside the square brackets:

$$F = 175[\frac{1((1.00295)^{60} - 1)}{1.00295 - 1}]$$

=> $F = 175[65.53543852]$

$= €11468.70$

6) Amortisation Formula/Pension Planning:

a) Terminology:

- **Annuity:** a regular stream of fixed payments over a specified time period of time, taking into account the time value of money. It is sometimes used in relation to regular pension payments that lasts as long as the person is alive.
- **Amortisation:** the process of accounting for a sum of money by making it equivalent to a series of payments over time.
- **Amortised Loan:** a loan that involves paying back a fixed amount at regular intervals over a fixed period of time e.g. mortgages or term loans

b) Amortisation Formula:

Notes:

- Need to know the proof (See Formal Proofs Pg 64)
- Can be used for a series of regular repayments:

$$A = \frac{P(i(1+i)^t)}{(1+i)^t - 1}$$

See Tables pg 31

Example: Daragh borrows €10,000 at an APR of 6%. He wants to repay it in five equal instalments over five years, with the first repayment one year after he takes out the loan. How much should each repayment be?

- We start by letting 'A' represent the repayments.
- That means the present value of the first repayment will be: $\frac{A}{1.06}$

- We now add together the present value of all future repayments, which gives:

$$\frac{A}{1.06} + \frac{A}{(1.06)^2} + \cdots + \frac{A}{(1.06)^5}$$

- A Geometric Series with a = $\frac{A}{1.06}$ and r = $\frac{1}{1.06}$.
- We can now fill our expressions for 'a' and 'r' into the formula for summing the terms of a Geometric Series:

$$S_n = \frac{\frac{A}{1.06}\left(1 - \left(\frac{1}{1.06}\right)^n\right)}{\frac{1}{1.06} - 1}$$

- The series above has 5 terms so we will now find S_5:

$$S_5 = \frac{\frac{A}{1.06}\left(1 - \left(\frac{1}{1.06}\right)^5\right)}{\frac{1}{1.06} - 1} = \frac{\frac{A}{1.06}(0.2527418271)}{-0.0566}$$

$$= \frac{A}{1.06}(4.465403306) = (4.212644628)A$$

- This value must equal the value of the loan now so:

$$4.212644628A = 10000$$

$$\Rightarrow A = \frac{10000}{4.212644628} \Rightarrow A = €2373.96$$

Note: We could have used the Amortisation Formula in this particular example:

$$A = P\frac{i(1+i)^t}{(1+i)^t - 1}$$

$$A = 10000\frac{0.06(1+0.06)^5}{(1+1.06)^5 - 1} = €2373.96$$

d) Pension Planning:

Example: Peter is 30 years old and is planning a pension for himself. He intends retiring in 35 years' time and wants to have a fund that could give him a payment of €20,000 at the start of each year for 25 years, from the date of his retirement. He assumes an annual growth rate of 4%.

i) Calculate the value on the date of retirement of the fund required.

ii) He plans to invest a fixed amount of money every month in order to generate the fund required. If he starts his payments immediately, what will his monthly payment have to be?

i) To pay himself €20,000 at the start of each year for 25 years, we firstly need to bring all future payments back to present values.

- The present values of these payments will be:

Retirement Fund = $\frac{20000}{(1.04)^0} + \frac{20000}{(1.04)^1} + \cdots + \frac{20000}{(1.04)^{24}}$

- A Geometric Series with a = 20000 and r = $\frac{1}{1.03}$ so:

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$\Rightarrow S_{25} = \frac{25000[1 - (\frac{1}{1.04})^{25}]}{1 - \frac{1}{1.04}}$$

$$\Rightarrow S_{25} = €406{,}174.08.$$

ii) Again, we can change the AER to a monthly rate, or make sure the time is in units of years.

- We will convert the AER in this example as in Section 5(a) above:

$$\Rightarrow 1.04 = 1(1+i)^{12}$$

$$\Rightarrow i = 0.00327374 \Rightarrow i = 0.3274\%$$

- Let P be the monthly deposit required to make a fund of €406,174.08 in 35 years' time.
- Payments start immediately, so the first payment will be in the account for the full 35 years, or 420 months and the next payment for 419 months etc.
- Last payment will be in the account for 1 month also:

$$F = P(1.003274)^{420} + P(1.003274)^{419} + \cdots + P(1.003274)^1$$

$$\Rightarrow 406{,}174.08 = P[(1.003274)^1 + \cdots + (1.003274)^{420}]$$

- The Geometric Series in the square brackets has a = $(1.003274)^1$ and r = 1.003274, so:

$$\Rightarrow 406{,}174.08 = P\left[\frac{1.003274((1.003274)^{420} - 1)}{1.003274 - 1}\right]$$

$$\Rightarrow 406{,}174.08 = P[902.9217431]$$

$$\Rightarrow P = \frac{406{,}174.08}{902.9217431} = €449.84$$