Past Exam Questions: The Line Week 20 revision

(a) $[AB]$ is a line segment.
The point $C\ (6, 11)$ divides the line segment $[AB]$ internally in the ratio $1:3$.

A is the point $(1, 13)$.
Find the co-ordinates of the point B.

$B = (x_2, y_2),\ m:n = 1:3$

$(6, 11) = \left(\frac{1(x_2)+3(1)}{1+3}, \frac{1(y_2)+3(13)}{1+3}\right)$

$6 = \frac{x_2+3}{4}$ and $11 = \frac{y_2+39}{4}$

$x_2 = 21$ and $y_2 = 5$

$B = (21, 5)$

(b) Find the perpendicular distance from the point $(5, -2)$ to the line:

$$y = \frac{4}{3}x - 11$$

$\frac{4}{3}x - y - 11 = 0$

Perpendicular distance:

$= \frac{\left|\frac{4}{3}(5) + (-1)(-2) - 11\right|}{\sqrt{\left(\frac{4}{3}\right)^2 + (-1)^2}}$

$= \frac{|-7|}{5}$

$= 1{\cdot}4$ [units]

(a) Find the area of the triangle with vertices $(4, 6)$, $(-3, -1)$, and $(0, 11)$.

$(4, 6), (-3, -1), (0, 11).$

-11

$(4, -5), (-3, -12), (0, 0)$

AREA $= \frac{1}{2}|4(-12) - (-3)(-5)|$

$= \frac{1}{2}|-63|$

$= 31 \cdot 5$

(a) The points $A\ (8, -4)$ and $B\ (-1, 3)$ are the endpoints of the line segment $[AB]$.

Find the coordinates of the point C, which divides $[AB]$ internally in the ratio $4:1$.

$C = \left(\frac{1(8)+4(-1)}{4+1}, \frac{1(-4)+4(3)}{4+1}\right)$

$C = \left(\frac{4}{5}, \frac{8}{5}\right)$

(b) $A(-1, k)$ and $B(5, l)$ are two points, where $k, l \in \mathbb{Q}$.

(i) Show that the midpoint of $[AB]$ is $\left(2, \frac{k+l}{2}\right)$.

Mid-point $= \left(\frac{-1+5}{2}, \frac{k+l}{2}\right)$

$= \left(2, \frac{k+l}{2}\right)$

OR

-1 to 5 is 6 steps, then $x = -1 + 3 = 2$

k to l is $(l-k)$ steps, then $y = k + \frac{l-k}{2} = \frac{k+l}{2}$

Mid-point $= \left(2, \frac{k+l}{2}\right)$

(ii) The perpendicular bisector of $[AB]$ is:

$$3x + 2y - 14 = 0$$

Find the value of l and the value of k.

Slope $AB = \frac{l-k}{5-(-1)} = \frac{l-k}{6}$

Perpendicular slope $= -\frac{6}{l-k}$

Slope of $3x + 2y - 14 = 0$ is $-\frac{3}{2}$

$-\frac{6}{l-k} = -\frac{3}{2}$ so $\boldsymbol{l - k = 4}$ ... Eqn 1

or

Slope $AB = \frac{2}{3}$, then $(-1, k)$ and $(5, l) \in$

$y = mx + c$, also gives $\boldsymbol{l - k = 4}$ ... Eqn 1

$\left(2, \frac{k+l}{2}\right) \to 3(2) + 2\left(\frac{k+l}{2}\right) - 14 = 0$

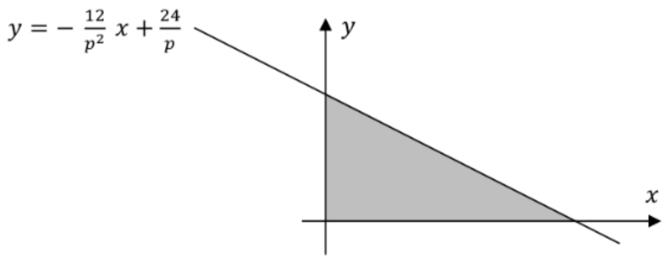
$\boldsymbol{k + l = 8}$ ... Eqn 2

$\begin{cases} l + k = 8 \\ l - k = 4 \end{cases}$

$2l = 12$ ..... $l = 6$, $k = 2$

(ii) The **area of the triangle** formed by the x-axis, the y-axis, and the tangent $y = -\frac{12}{p^2}x + \frac{24}{p}$ is always k square units, where $k \in \mathbb{N}$ is a constant.

Work out the value of k.



<u>y-intercept:</u>

$y = -\frac{12}{p^2}(0) + \frac{24}{p}$

$y = \frac{24}{p}$

$\left(0, \frac{24}{p}\right)$

height $= \frac{24}{p}$

<u>x-intercept:</u>

$0 = -\frac{12}{p^2}x + \frac{24}{p}$

$\frac{12}{p^2}x = \frac{24}{p}$

$\frac{12}{p}x = 24$

$x = 2p$

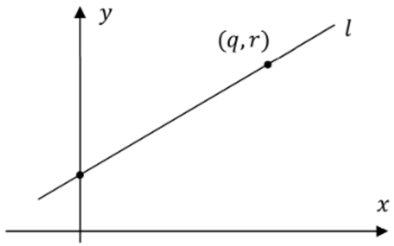
$(2p, 0)$

base $= 2p$

Area $= \frac{1}{2}(2p)\left(\frac{24}{p}\right)$

$= 24$ units^2

(b) The line l has a slope of m and contains the point (q, r), where $m, q, r \in \mathbb{R}$ are all positive. Find the co-ordinates of the point where l cuts the y-axis, in terms of m, q, and r.



$$y - y_1 = m(x - x_1)$$
$$y - r = m(x - q)$$
$$y - r = mx - mq$$
$$y = mx - mq + r$$
$x = 0$, so $y = -mq + r$

Answer: $(0, -mq + r)$

(a) The line $3x - 6y + 2 = 0$ contains the point $\left(k, \frac{2k+2}{3}\right)$, where $k \in \mathbb{R}$. Find the value of k.

$$3k - 6\left(\frac{2k+2}{3}\right) + 2 = 0$$
$$\Rightarrow 3k - 4k - 4 + 2 = 0$$
$$\Rightarrow k = -2$$

(c) The line k has a slope of -2. The line j makes an angle of $30°$ with k.

Find **one** possible value of the slope of the line j. Give your answer in the form $d + e\sqrt{f}$, where $d, e, f \in \mathbb{Z}$.

$$\tan 30° = \frac{-2 - m_2}{1 + (-2)m_2}$$

So $\frac{1}{\sqrt{3}} = \frac{-2 - m_2}{1 - 2m_2}$

$$1 - 2m_2 = -2\sqrt{3} - \sqrt{3}m_2$$
$$m_2 = \frac{1 + 2\sqrt{3}}{2 - \sqrt{3}}$$
$$m_2 = \frac{1 + 2\sqrt{3}}{2 - \sqrt{3}} \times \frac{2 + \sqrt{3}}{2 + \sqrt{3}}$$
$$m_2 = 8 + 5\sqrt{3}$$

(a) The coordinates of three points are $A(2, -6)$, $B(6, -12)$, and $C(-4, 3)$. Find the perpendicular distance from A to BC.

Based on your answer, what can you conclude about the relationship between the points A, B, and C?

Slope of BC $m = \frac{3+12}{-4-6} = -\frac{3}{2}$

Equation BC $\quad 3x + 2y + 6 = 0.$

Perp. Distance from A to line BC

$$\frac{3(2) + 2(-6) + 6}{\sqrt{3^2 + 2^2}} = \frac{6 - 12 + 6}{\sqrt{13}} = \frac{0}{\sqrt{13}} = 0.$$

Therefore A, B and C are collinear.

(b) The point $P(s,t)$ is on the line $x - 2y - 8 = 0$.
The point P is also a distance of 1 unit from the line $4x + 3y + 6 = 0$.
Find a value of s and the corresponding value of t.

$s - 2t - 8 = 0$ so $s = 2t + 8$

$$\frac{|4s+3t+6|}{\sqrt{4^2+3^2}} = 1$$

$$\frac{|8t+32+3t+6|}{5} = 1$$

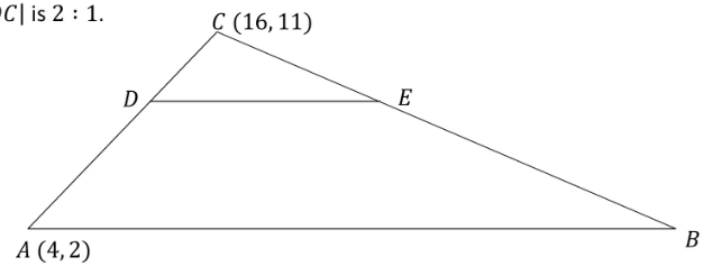
$|11t + 38| = 5$

$11t + 38 = 5$ **or** $11t + 38 = -5$

$\therefore\ t = -3$ **or** $\therefore\ t = -\frac{43}{11}$

$\therefore\ s = 2$ **or** $\therefore\ s = \frac{2}{11}$

(c) The points $A(4,2)$ and $C(16,11)$ are vertices of the triangle ABC shown below.
D and E are points on $[CA]$ and $[CB]$ respectively.
The ratio $|AD| : |DC|$ is $2 : 1$.



(i) Find $|AD|$.

$$\left(\frac{2\times 16 + 1\times 4}{2+1}, \frac{2\times 11 + 1\times 2}{2+1}\right)$$

$= D(12,8)$

$|AD| = \sqrt{(12-8)^2 + (8-2)^2} = 10$

(ii) $[AB]$ and $[DE]$ are **horizontal** line segments.
$|AB| = 33$ units.
Find the coordinates of B and of E.

$|AB| = 33 \Rightarrow B\ is\ (37,2)$.

The translation $\overrightarrow{CB}$: x increases by 21

$x_E = 16 + \frac{1}{3}(21) = 23$

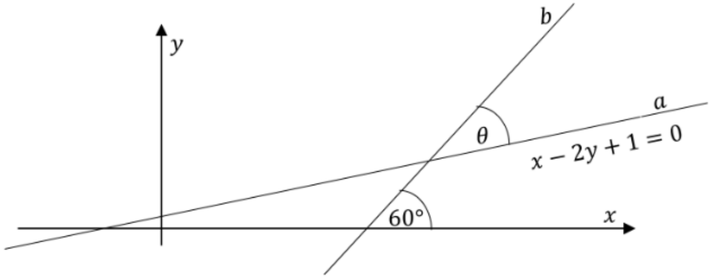
The translation $\overrightarrow{CB}$: y decreases by 9

$y_E = 11 - \frac{1}{3}(9) = 8$

So $E = (23,8)$.

$B($, $)$ $E($, $)$

(b) The diagram below shows two lines a and b. The equation of a is $x - 2y + 1 = 0$.
The acute angle between a and b is θ. Line b makes an angle of $60°$ with the positive sense of the x-axis, as shown in the diagram.
Find the value of θ, in degrees, correct to 3 decimal places.



Slope of $a = \frac{1}{2}$

Slope of $b = \tan 60° = \sqrt{3}$

$$\tan\theta = \pm\frac{\sqrt{3} - \frac{1}{2}}{1 + \frac{\sqrt{3}}{2}} = \pm\frac{2\sqrt{3} - 1}{2 + \sqrt{3}}$$

$$= \pm\frac{(2\sqrt{3} - 1)(2 - \sqrt{3})}{(2 + \sqrt{3})(2 - \sqrt{3})}$$

$$= \pm(-8 + 5\sqrt{3})$$

$$\theta = \tan^{-1}(-8 + 5\sqrt{3})$$

$$\theta = 33{\cdot}435°$$

(b) The line l has a slope m, and contains the point $A(6, 0)$.

(i) Write the equation of the line l in terms of m.

$y - 0 = m(x - 6)$ <u>or</u> $y = m(x - 6)$

Or

$$y = mx - 6m$$

Or

$$y = mx + c$$

$\therefore 0 = 6m + c \Rightarrow c = -6m$

(ii) The line l cuts the line k: $4x + 3y = 25$ at P.
Find the co-ordinates of P in terms of m.
Give each co-ordinate as a fraction in its simplest form.

$$y = m(x - 6)$$

$$4x + 3y = 25$$

$\Rightarrow 4x + 3m(x - 6) = 25$

$\Rightarrow x = \frac{25+18m}{3m+4}$

Substitute this into $y = m(x - 6)$

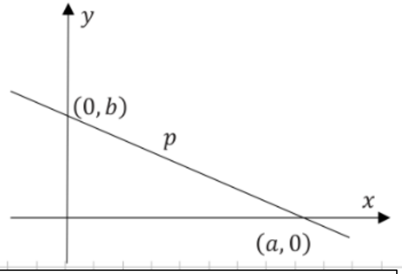
$$y = m\left(\frac{25 + 18m}{3m + 4}\right) - 6m$$

$$= \frac{25m + 18m^2 - 18m^2 - 24m}{3m + 4}$$

$$= \frac{m}{3m + 4}$$

(a) The line p makes an intercept on the x-axis at $(a, 0)$ and on the y-axis at $(0, b)$, where $a, b \neq 0$.

Show that the equation of p can be written as $\frac{x}{a} + \frac{y}{b} = 1$.



$m = \frac{b-0}{0-a} = \frac{-b}{a}$

$y - 0 = \frac{-b}{a}(x - a)$

$ay = -bx + ab$

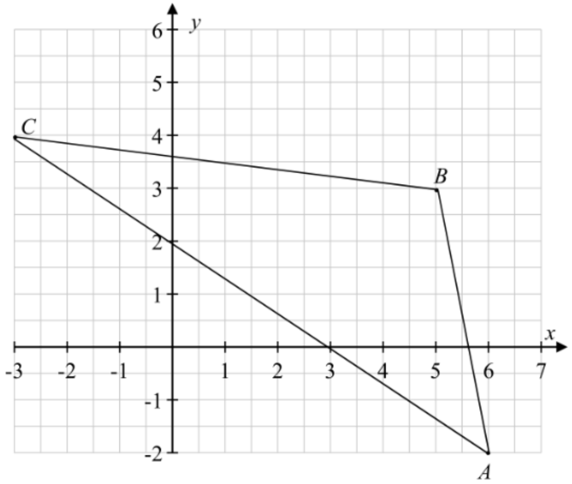
$bx + ay = ab$

Now divide across by ab

$$\frac{x}{a} + \frac{y}{b} = 1$$

The points $A(6, -2)$, $B(5, 3)$ and $C(-3, 4)$ are shown on the diagram.

(a) Find the equation of the line through B which is perpendicular to AC.



Slope $AC = -\frac{2}{3}$

perp. slope $= \frac{3}{2}$

$y - 3 = \frac{3}{2}(x - 5)$

$3x - 2y = 9$

(b) Use your answer to part **(a)** above to find the co-ordinates of the orthocentre of the triangle ABC.

Point of intersection of the altitudes

Slope $AB = \frac{3+2}{5-6} = -\frac{5}{1}$

perp. slope $= \frac{1}{5}$

$y - 4 = \frac{1}{5}(x + 3)$

$x - 5y + 23 = 0$

Orthocentre:

$3x - 2y = 9 \cap x - 5y = -23$

$\Rightarrow y = 6 \quad x = 7$

$(7, 6)$

The line $m: 2x + 3y + 1 = 0$ is parallel to the line $n: 2x + 3y - 51 = 0$.

(a) Verify that $A(-2, 1)$ is on m.

$2(-2) + 3(1) + 1 = 0$

or $-4 + 3 + 1 = 0$

(b) Find the coordinates of B, the point on the line n closest to A, as shown below.

Slope of m or $n = \frac{-2}{3}$

Slope of AB is $\frac{3}{2}$ and $(-2, 1)$ is on AB

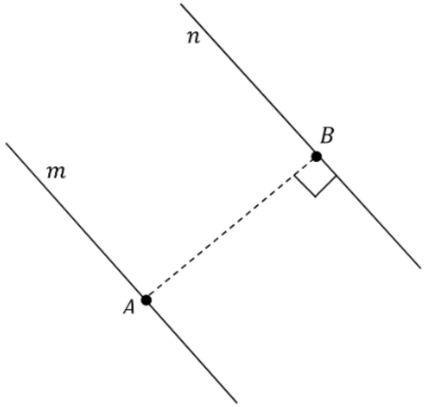
$$y - 1 = \frac{3}{2}(x - (-2))$$

equation of AB is $3x - 2y + 8 = 0$

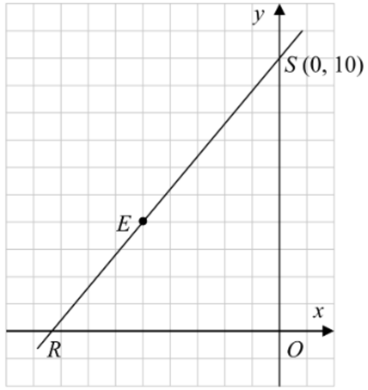
Solve for (x, y) between

$3x - 2y + 8 = 0$ and $2x + 3y - 51 = 0$

$n \cap AB = (6, 13) = B$



The line RS cuts the x-axis at the point R and the y-axis at the point $S(0, 10)$, as shown. The area of the triangle ROS, where O is the origin, is $\frac{125}{3}$.



(a) Find the co-ordinates of R.

Area $ROS = \frac{1}{2}|RO|.|OS| = \frac{125}{3}$

$\Rightarrow \frac{1}{2}|RO|(10) = \frac{125}{3}$

$\Rightarrow |RO| = \frac{25}{3}$

$R\left(-\frac{25}{3}, 0\right)$

(b) Show that the point $E(-5, 4)$ is on the line RS.

Slope RS $= \frac{10-0}{0+\frac{25}{3}} = \frac{6}{5}$ Slope ES $= \frac{10-4}{0+5} = \frac{6}{5}$ Slope ER $= \frac{4-0}{-5+\frac{25}{3}} = \frac{6}{5}$

Any two slopes correct $\Rightarrow (-5, 4) \in RS$

(c) A second line $y = mx + c$, where m and c are positive constants, passes through the point E and again makes a triangle of area $\frac{125}{3}$ with the axes. Find the value of m and the value of c.

$y = mx + c$ cuts x-axis at $P\left(-\frac{c}{m}, 0\right)$ and cuts y-axis at Q(0, c)

Area $\Delta POQ = \frac{1}{2}\left|0 - \left(-\frac{c}{m}\right)c\right| = \frac{1}{2}\left|\frac{c^2}{m}\right| = \frac{125}{3} \Rightarrow m = \frac{3c^2}{250}$

$(-5, 4) \in y = mx + c \Rightarrow 4 = -5m + c \Rightarrow 4 = -5\left(\frac{3c^2}{250}\right) + c \Rightarrow 3c^2 - 50c + 200 = 0$

$\Rightarrow (3c - 20)(c - 10) = 0 \Rightarrow c = \frac{20}{3}$ or $c = 10$

$c = \frac{20}{3}$

Hence, $m = \frac{3c^2}{250} = \frac{3(\frac{20}{3})^2}{250} = \frac{400}{750} = \frac{8}{15}$

(a) The co-ordinates of two points are $A(4,\ -1)$ and $B(7,\ t)$.

The line $l_1 : 3x - 4y - 12 = 0$ is perpendicular to AB. Find the value of t.

Slope $AB = \dfrac{t+1}{7-4} = \dfrac{t+1}{3}$ Slope $l_1 = \dfrac{3}{4}$

$AB \perp l_1 \Rightarrow \dfrac{t+1}{3} \times \dfrac{3}{4} = -1 \Rightarrow t + 1 = -4 \Rightarrow t = -5$

(b) Find, in terms of k, the distance between the point $P(10,\ k)$ and l_1.

$$d = \left| \frac{3(10) - 4k - 12}{\sqrt{3^2 + 4^2}} \right| = \left| \frac{18 - 4k}{5} \right|$$

(c) $P(10,\ k)$ is on a bisector of the angles between the lines l_1 and $l_2 : 5x + 12y - 20 = 0$.

(i) Find the possible values of k.

$$\left| \frac{18-4k}{5} \right| = \left| \frac{50+12k-20}{\sqrt{5^2+12^2}} \right|$$

$$\Rightarrow \left| \frac{18-4k}{5} \right| = \left| \frac{30+12k}{13} \right|$$

$$\Rightarrow 13(18-4k) = \pm 5(30+12k)$$

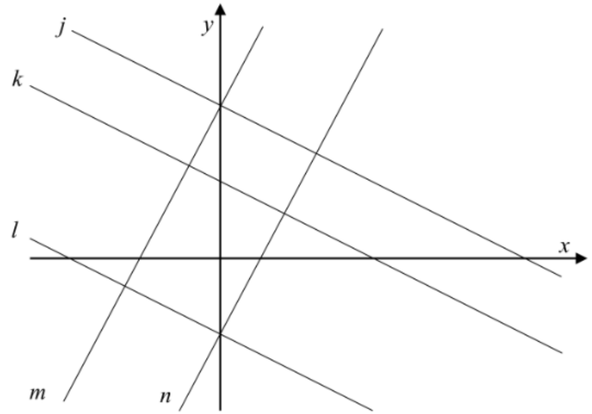
$$\Rightarrow -112k = -84 \quad \text{or} \quad 8k = -384$$

$$\Rightarrow k = \tfrac{3}{4} \quad \text{or} \quad k = -48$$

(ii) If $k > 0$, find the distance from P to l_1.

$$k = \frac{3}{4} \Rightarrow d = \left| \frac{18 - 4\left(\frac{3}{4}\right)}{5} \right| = 3$$

In the co-ordinate diagram shown, the lines j, k, and l are parallel, and so are the lines m and n. The equations of four of the five lines are given in the table below.



Equation	Line
$x + 2y = -4$	
$2x - y = -4$	
$x + 2y = 8$	
$2x - y = 2$	

$x + 2y = -4 \Rightarrow \quad y = -\frac{1}{2}x - 2 \quad \rightarrow l$

$2x - y = -4 \Rightarrow \quad y = 2x + 4 \quad \rightarrow m$

$x + 2y = 8 \Rightarrow \quad y = -\frac{1}{2}x + 4 \quad \rightarrow j$

$2x - y = 2 \Rightarrow \quad y = 2x - 2 \quad \rightarrow n$

(a) Complete the table, by matching four of the lines to their equations.

(b) Hence, insert scales on the x-axis and y-axis.

(c) Hence, find the equation of the remaining line, given that its x-intercept and y-intercept are both integers.

Equation of k: $y = -\frac{1}{2}x + 2$

or

$x + 2y = 4$

The equations of six lines are given:

Line	Equation
h	$x = 3 - y$
i	$2x - 4y = 3$
k	$y = -\frac{1}{4}(2x - 7)$
l	$4x - 2y - 5 = 0$
m	$x + \sqrt{3}y - 10 = 0$
n	$\sqrt{3}x + y - 10 = 0$

Description	Line(s)
A line with a slope of 2.	l
A line which intersects the y-axis at $(0, -2\frac{1}{2})$.	l
A line which makes equal intercepts on the axes.	h
A line which makes an angle of 150° with the positive sense of the x-axis.	m
Two lines which are perpendicular to each other.	l and k

(b) Find the acute angle between the lines m and n.

Slope of m: $m_1 = -\frac{1}{\sqrt{3}}$

Slope of n: $m_2 = -\sqrt{3}$

$$\tan\theta = \pm\frac{m_1 - m_2}{1 + m_1 m_2} = \pm\frac{-\frac{1}{\sqrt{3}} + \sqrt{3}}{1 - \frac{1}{\sqrt{3}}\left(-\sqrt{3}\right)} = \pm\frac{\frac{-1+3}{\sqrt{3}}}{1+1} = \pm\frac{1}{\sqrt{3}}$$

$$\tan\theta = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad \theta = 30^\circ$$