- Chapter 9: Differentiation And Integration
- Topic 37: Recap on Logarithms

- Definition of Logs

- You might recall learning about logarithms on your Maths course, so we need to start by familiarising ourselves with them before starting this topic.
- Logarithms are basically the opposite of raising a number to a particular power.
- We will begin with a definition of what logs are:

$$\boxed{a^m = n \Leftrightarrow \log_a n = m}$$

- $a^m = n$ is called "index form"
- 'a' is called the "base" and $a > 0$
- We say, "log of n to the base a is m".

- For example:

  If $2^5 = 32$, then we can write that using logs by writing: $\log_2 32 = 5$.

- Laws of Logs:

- Recall that we have a set of Laws we have to follow when dealing with Logs.
- There are 8 laws as laid out below, but the main ones we will need for this course are laws 1, 2, 3, 6 and 7.
- These laws are on pg21 of the Formulae Book.

Law 1:	$\log_a(xy) = \log_a x + \log_a y$	Example:	$\log_2(20) = \log_2 5 + \log_2 4$
Law 2:	$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$	Example:	$\log_5\left(\frac{2}{3}\right) = \log_5 2 - \log_5 3$
Law 3:	$\log_a(x^p) = p\log_a x$	Example:	$\log_2 x^3 = 3\log_2 x$
Law 4:	$\log_a 1 = 0$	Examples:	$\log_4 1 = 0$
Law 5:	$\log_a\left(\frac{1}{x}\right) = -\log_a x$	Example:	$\log_2\left(\frac{1}{8}\right) = -\log_2 8$
Law 6:	$\log_a(a^x) = x$	Example:	$\log_4(4^x) = x$
Law 7:	$a^{\log_a x} = x$	Example:	$2^{\log_2 x} = x$
Law 8: (Change of base law)	$\log_b x = \frac{\log_a x}{\log_a b}$	Examples:	$\log_8 x = \frac{\log_2 x}{\log_2 8}$

- Example 1: Simplify $\log_3 2 + 2\log_3 3 - \log_3 18$.

  Solution:

  Method 1:

  $\log_3 2 + 2\log_3 3 - \log_3 18$
  $= \log_3 2 + 2(1) - [\log_3 9 + \log_3 2]$ (Using Law 6 for $\log_3 3$ and Law 1 for $\log_3 18$)
  $= \log_3 2 + 2 - \log_3 9 - \log_3 2$ (removing the square brackets)
  $= 2 - \log_3 9$ (as $\log_3 2 - \log_3 2 = 0$)
  $= 2 - 2$ (as $\log_3 9 = 2$)
  $= 0$

  Method 2:

  $\log_3 2 + 2\log_3 3 - \log_3 18$
  $= \log_3 2 + \log_3 3^2 - \log_3 18$ (Using Law 3)
  $= \log_3 2 + \log_3 9 - \log_3 18$
  $= \log_3(2)(9) - \log_3 18$ (Putting $\log_3 2 + \log_3 9$ together using Law 1)
  $= \log_3 18 - \log_3 18$
  $= 0$

- Solving Log Equations:

- When solving equations with logs, we have a rule we can use, similar to the one we used when solving indices equations:

If $\log_a b = \log_a c$, then b = c. → N.B. Bases must be the same.

- Example 3: Solve the equation $\log_3(x+1) = \log_3(x-1)^2$.

Solution:

- As the base is the same on both sides of this equation, and we have a single log on both sides, we can simply equate the (x + 1) and $(x - 1)^2$:

  $x + 1 = (x - 1)^2$
  $\Rightarrow x + 1 = x^2 - 2x + 1$
  $\Rightarrow x^2 - 3x = 0$
  $\Rightarrow x(x - 3) = 0$
  $\Rightarrow x = 0$ or $x - 3 = 0$
  $\Rightarrow x = 0$ or $x = 3$

- We always have to check all our answers when solving log equations in case one, or both of them, leads to a negative log:

If x = 0:	If x = 3:
$\log_3(0+1) = \log_3(0-1)^2$	$\log_3(3+1) = \log_3(3-1)^2$
$\log_3(1) = \log_3(-1)^2$	$\log_3(4) = \log_3(2)^2$
$\log_3 1 = \log_3 1$	$\log_3 4 = \log_3 4$

- So, in this case, both answers are correct i.e. x = 0, or x = 3.

Important Note:

- If a log is to the base e, we call it a natural log and write it as "ln".
  - For example, ln 5 means $\log_e 5$
- We treat it the same as all other logs and it follows the Laws of Logs as usual.

- Example 4: Solve the equation $\log_3(x+1) - \log_3(x-1) = 1$.

Solution:

- The aim here is to tidy up both sides into a single log, like in Example 2 above, so that we can eliminate the logs:
- We use Law 2 to tidy up the LHS, and Law 6 to rewrite the 1 on the RHS as a log:

$\log_3(x+1) - \log_3(x-1) = 1$

=> $\log_3 \frac{x+1}{x-1} = \log_3 3$

=> $\frac{x+1}{x-1} = 3$

=> $x + 1 = 3(\text{x} - 1)$

=> $x + 1 = 3\text{x} - 3$

=> $2x = 4$

=> $x = 2$

Check:

$\log_3(x+1) - \log_3(x-1) = 1$

=> $\log_3(2+1) - \log_3(2-1) = 1$

=> $\log_3(3) - \log_3(1) = 1$

=> $1 - 0 = 1$

=> $1 = 1$

Classwork Questions: Pg 166 Ex 9A A selection from Qs 1/2

Topic 38: Rules of Integration

- Mathematically, Integration is the reverse process of Differentiating a function.
- Our first basic rule of Differentiation was "Multiply by the power and reduce the power by one", so our first basic rule of Integration is "**Add one to the power and divide by the new power**", or symbolically:

$$\int x^n . dx = \frac{x^{n+1}}{n+1} + c$$

1. $\int$ is the symbol we use for integrating
2. 'dx' tells us what variable we are integrating by
3. 'c' is a constant (See below)

- Why we need the constant 'c'?
- If we differentiated the functions $f(x) = x^2 + 5$ and $g(x) = x^2 - 3$, we would get $f'(x) = 2x$ and $g'(x) = 2x$.
- You can see that the derivatives turn out the exact same even though the two functions $f(x)$ and $g(x)$ are not the same.
- To sort this, the constant of integration 'c' is used, and you only need to add it to the end, when you are finished integrating the expression.

- Examples: Evaluate the following: i) $\int x^2\,dx$ ii) $\int 4x^3 + 4\,dx$ iii) $\int \frac{1}{x^3} dx$ iv) $\int \frac{1}{\sqrt{x}} dx$ v) $\int \frac{4 + x^2}{\sqrt{x}} dx$

Solution:

i) Following our basic rule above gives:

$$\int x^2\,dx = \frac{x^{2+1}}{3} = \frac{x^3}{3} + c$$

This solution is called the Integral of x^2 or the Antiderivative of x^2.

ii) Before integrating this expression, we will rewrite the constant 3 as $3x^0$, to see what happens to a constant when we integrate it.

$$\int 4x^3 + 3\,dx$$
$$= \int 4x^3 + 3x^0\,dx$$
$$= 4\left(\frac{x^{3+1}}{4}\right) + 3\left(\frac{x^{0+1}}{1}\right)$$
$$= x^4 + 3x + c$$

Notes:

1) We can keep the constants '3' and '4' out in front as they don't affect the integral. We just multiply them back in at the end.
2) The shortcut for integrating a constant with respect to x is to just multiply it by an 'x' e.g. $\int 6\,.dx = 6x$

iii) We will use a law of Indices first to bring the x^3 above the line:

$$\int \frac{1}{x^3} dx = \int x^{-3}\,dx$$
$$= \frac{x^{-3+1}}{-2}$$
$$= \frac{x^{-2}}{-2} = \frac{1}{-2x^2} + c$$

iv) Again, we will rewrite the expression first before integrating:

$$\int \frac{1}{\sqrt{x}} dx = \int \frac{1}{x^{\frac{1}{2}}} dx = \int x^{-\frac{1}{2}} dx$$

$$= \frac{x^{-\frac{1}{2}+1}}{\frac{1}{2}}$$

$$= \frac{x^{\frac{1}{2}}}{\frac{1}{2}}$$

$$= 2\sqrt{x} + c$$

v) We will break up the fraction first:

$$\int \frac{4 + x^2}{\sqrt{x}} dx = \int \frac{4}{\sqrt{x}} + \frac{x^2}{\sqrt{x}} dx = \int 4x^{-\frac{1}{2}} + x^{2-\frac{1}{2}} = \int 4x^{-\frac{1}{2}} + x^{\frac{3}{2}}$$

$$= 4\left(\frac{x^{-\frac{1}{2}+1}}{\frac{1}{2}}\right) + \frac{x^{\frac{3}{2}+1}}{\frac{5}{2}}$$

$$= 8\left(\frac{x^{\frac{1}{2}}}{1}\right) + \frac{2}{5}\left(\frac{x^{\frac{5}{2}}}{1}\right)$$

$$= 8\sqrt{x} + \frac{2x^{\frac{5}{2}}}{5} + c$$

- Integral of ${}^{1}/_{x}$:

- What about $\int \frac{1}{x} dx$?
- If we follow the rule of integration from above, the following happens:

$$\int \frac{1}{x} dx = \int x^{-1}\, dx = \frac{x^{-1+1}}{0}$$

- We know from the Differentiation chapter that the derivative of $\ln x$ is $\frac{1}{x}$, so it follows that the antiderivative of $\frac{1}{x}$ is $\ln x$, so:

$$\boxed{\int \frac{1}{x}.\, dx = \ln x + c}$$

See Tables pg26

$$\boxed{\int \frac{1}{ax+b}.\, dx = \frac{1}{a}\ln(ax+b) + c}$$

Not in Tables

- Example: Evaluate the following: $\int \frac{1}{3x-7} dx$.

Solution:

$$\int \frac{1}{ax+b}.\, dx = \frac{1}{a}\ln(ax+b) + c$$

$$\Rightarrow \int \frac{1}{3x-7}.\, dx = \frac{1}{3}\ln(3x-7) + c$$

- <u>Integral of Trig/Exponential Functions:</u>

- When differentiating trigonometric and exponential functions, we used the log tables.
- We do the same when integrating.

$f(x)$	$\int f(x)dx$
$x^n \quad (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln\|x\|$
e^x	e^x
e^{ax}	$\frac{1}{a}e^{ax}$
a^x	$\frac{a^x}{\ln a}$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\tan x$	$\ln\|\sec x\|$

See Tables pg 26

NOT in Tables

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax + c$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax + c$$

- **Examples:** Evaluate the following: i) $\int e^{3x}\, dx$ ii) $\int e^{-4x}\, dx$ iii) $\int 3^x\, dx$ iv) $\int sin5x\, dx$ v) $\int sin3x + cos4x\, dx$

<u>Solution:</u>

i) Using the rule for e^{ax} from the table above:

$$\int e^{3x}\, dx = \frac{1}{3}e^{3x} + c$$

ii) Using the same rule as part (i):

$$\int e^{-4x}\, dx = -\frac{1}{4}e^{-4x} + c$$

iii) We will have to use the rule for a^x for this one:

$$\int 3^x\, dx = \frac{3^x}{ln3} + c$$

iv) When integrating the sine function, we normally use the table above, but in this case the angle is 5x, so we have to use the related rule highlighted in yellow above:

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax + c$$

$$\int \sin 5x\, dx = -\frac{1}{5}\cos 5x + c$$

v) In this question, we just handle each trig function separately, we did before:

$$\int sin3x + cos4x\, dx = -\frac{1}{3}\cos 3x + \frac{1}{4}\sin 4x + c$$

- Two further Integrals:

- There are two final rules from the Tables that may arise on our course, that wouldn't have been covered on your Maths course:

$f(x)$	$\int f(x)dx$
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\dfrac{x}{a}$
$\dfrac{1}{x^2+a^2}$	$\dfrac{1}{a}\tan^{-1}\dfrac{x}{a}$

See Tables pg 26

- Example: Evaluate the following: $\int \frac{1}{x^2+16}dx$

Solution:

- Before applying the rule above, we have to re-write the denominator in the form $x^2 + a^2$:

$$\int \frac{1}{x^2 + 16}dx = \int \frac{1}{x^2 + 4^2}dx = \frac{1}{4}\tan^{-1}\frac{x}{4} + c$$

Classwork Questions: Pg 167 Ex 9B A selection from Qs 1 – 10

- Definite Integrals

- Sometimes we have given bounds or limits on an integral and when this happens, we don't have a constant of integration.
- In these cases, we can evaluate the definite integral by filling in the limits to our solution and following the rule below:

$$\boxed{\int_a^b f(x).dx = F(x)_a^b = F(b) - F(a)}$$

1. $F(x)$ is the integral of f(x)

2. The order of the limits is important i.e. it's always F(top limit) – F(bottom limit)

- Example: Evaluate $\int_{-2}^{2} x(x+4)\,dx$.

Solution:

- First, we integrate the function as before, taking care to get rid of the brackets first:

$$\int_{-2}^{2} x(x+4)\,dx = \int_{-2}^{2} x^2 + 4x\,dx = \frac{x^3}{3} + \frac{4x^2}{2} = \frac{x^3}{3} + 2x^2$$

- We now evaluate the integral between the two limits using the rule above:

$$F(2) - F(-2) = \frac{(2)^3}{3} + 2(2)^2 - [\frac{(-2)^3}{3} + 2(-2)^2]$$
$$= \frac{8}{3} + 8 + \frac{8}{3} - 8 = \frac{16}{3}$$

Classwork Questions: Pg 168/169 Ex 9C A selection from Qs 1 – 20

Topic 39: Intgeration by Parts

- When learning Differentiation you learned the Product Rule for differentiating functions that contain two functions multiplied by each other e.g.s $y = x.e^x$ or $y = x.sinx$
- When integrating functions like this, we use a method called integration by parts:

$$\boxed{\int u.dv = uv - \int v.du}$$

See pg26 Tables Book

- In order for this rule to work, we also follow the INLATE rule, to help us decide which part is the 'u' and which part is the 'dv'.
- We always begin by rearrange our functions in the following order before we start integrating them:

  In = Inverse Trig e.g. $\sin^{-1} x$
  L = Logs e.g. $\log x$
  A = Algebra e.g. 3x or x^3
  T = Trig e.g. $\sin x, \cos x$
  E = Exponential e.g. e^{4x}, e^{-x}

- **Example 1:** Find $\int x\, e^{2x} dx$.

  Solution:

- Firstly, we check that the functions are in the right order from the INLATE rule, which they are as the algebra 'x' is first followed by the exponential 'e^{2x}'.
- Now to integrate a function in the form $\int u.dv$ we split it into two separate calculations as shown below:

$u = x$ - On this side, we differentiate both sides with respect to the variable on the right: $\frac{d}{dx}(u) = \frac{d}{dx}(x)$ => $\frac{du}{dx} = 1$ => $du = dx$ (multiplying both sides by dx)	$dv = e^{2x}.dx$ - On this side, we integrate both sides to evaluate what 'v' is: $\int dv = \int e^{2x}.dx$ => $v = \frac{1}{2}e^{2x}$

- We now sub into our rule in the red box above:

$$\int u.dv = uv - \int v.du$$
$$=> \int x\,.e^{2x}.dx = (x)\left(\frac{1}{2}e^{2x}\right) - \int \frac{1}{2}e^{2x}\,.dx$$
$$=> \int x\,.e^{2x}.dx = \frac{1}{2}xe^{2x} - \frac{1}{2}\left(\frac{1}{2}e^{2x}\right) + c$$
$$=> \int x\,.e^{2x}.dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c$$

Note that we could check our answer by now differentiating this function.

Classwork Questions: Pg 170 Ex 9D Qs 1/2/6

- Now let's look at a slightly trickier one involving two iterations of what we were doing the last day.

- Example 2: Find $\int e^x \cos x\, dx$.

  Solution:

- This time the INLATE rule means that the two functions are in the wrong order as the Trig function (cos x) comes after the exponential (e^x), so we swap them around to:

$$\int \cos x\, e^x dx$$

- For a reason that will become clearer later on, we will give this integral a name……we will use 'A':

=> $A = \int \cos x\, e^x dx$

- Now to integrate A we split it into two separate calculations again as shown below:

$u = \cos x$	$dv = e^x . dx$
- On this side, we differentiate both sides with respect to the variable on the right: $\frac{d}{dx}(u) = \frac{d}{dx}(\cos x)$ => $\frac{du}{dx} = -\sin x$ => $du = -\sin x\, dx$	- On this side, we integrate both sides to evaluate what 'v' is: $\int dv = \int e^x . dx$ => $v = e^x$

- We now sub into our rule in the red box above:

$\int u.dv = uv - \int v.du$

=> $A = (\cos x)(e^x) - \int e^x . -\sin x\, dx$

=> $A = (\cos x)(e^x) + \int e^x \sin x\, dx$

=> $A = (\cos x)(e^x) + B$ where B = $\int e^x \sin x\, dx$

- Now you can see that we have created another integral B on the end, that we need to repeat the process for again.
- To integrate B, we follow the INLATE rule again and rearrange the functions into the correct order:

$u = \sin x$	$dv = e^x . dx$
- On this side, we differentiate both sides with respect to the variable on the right: $\frac{d}{dx}(u) = \frac{d}{dx}(\sin x)$ => $\frac{du}{dx} = \cos x$ => $du = \cos x\, dx$	- On this side, we integrate both sides to evaluate what 'v' is: $\int dv = \int e^x . dx$ => $v = e^x$

- Now subbing into our rule gives:

$\int u.dv = uv - \int v.du$

=> $B = (\sin x)(e^x) - \int e^x . \cos x\, dx$

=> $B = e^x \sin x - \int e^x . \cos x\, dx$

- You might now spot that the last integral is the exact same as A that we started out with, so there is a nice way to finish from here and prevent us going on indefinitely:

=> $A = (\cos x)(e^x) + e^x \sin x - A$

=> $2A = (\cos x)(e^x) + e^x \sin x$ (bringing the As to the LHS)

=> $A = \frac{1}{2}(e^x \cos x + e^x \sin x)$

=> $A = \frac{1}{2}e^x(\cos x + \sin x) + C$

Day 2: Classwork Questions: Pg 170 Ex 9D Qs 3/9/10/12/15

➢ Topic 40: Integrating Composite Functions

- When learning how to differentiate composite functions e.g. $(2x - 1)^3$, you learned about the Chain Rule.
- Unfortunately, when integrating there is no corresponding rule we can use, so we have to use a different technique called substitution.
- The aim is to choose our substitution cleverly in order to convert the composite function back to a more straight forward function from the Tables that we have rules for.

• Example 1: Pg 172 Ex 9E Q3

$$\int_0^1 \frac{3x^2}{x^3+2}.dx$$

Solution:

- The most important step is to choose our substitution correctly at the beginning.
- We have a few choices……

1) $u = 3x^2$ 2) $u = x^3 + 2$ 3) $u = \frac{3x^2}{x^3+2}$ (pointless as it's the entire original function)

- After some practice, you will hopefully be able to look ahead a few steps and see that the best choice is without having to try any workings.
- In this case, the best choice is number 2, the reason for which might become apparent later on.

$u = x^3 + 2$

=> $\frac{du}{dx} = 3x^2$

=> $du = 3x^2 dx$

Notice that we have created the function that is on top in the original question.

- So our original integral reduces to:

$$\int_0^1 \frac{3x^2}{x^3+2}.dx = \int_0^1 \frac{1}{u}.du$$

- We use the Tables to integrate $\frac{1}{u}$:

$$\int_0^1 \frac{1}{u}.du = \ln u$$

- We now have two choices that both yield the same answer:

Option 1: Change the original limits using ★

Option 2: Use ★ to replace u again at the end

Option 1: Change the limits	Option 2: Sub in for u
$u = x^3 + 2$ If x = 1 => $u = (1)^3 + 2 = 3$ If x = 0 => $u = (0)^3 + 2 = 2$ - Now we evaluate $\ln u$ between 2 and 3: $\ln(3) - \ln(2) = \ln(\frac{3}{2})$	$\ln u = \ln(x^3 + 2)$ - Now we evaluate this between 0 and 1: $= \ln((1)^3 + 2) - \ln((0)^3 + 2)$ $= \ln(3) - \ln(2) = \ln(\frac{3}{2})$

Classwork Questions: Pg 172 Ex 9E Qs 1/2/4/6/9

- Example 2: Pg 172 Ex 9E Q14

$$\int_1^2 \frac{x}{x+1}.dx$$

Solution:

- Again, we have two choices here……

  1) $u = x$ 2) $u = x + 1$

- Having seen the last example, maybe you can see why 1 would be a poor choice, so 2 is the best choice here.

$u = x + 1$ ★

=> $\frac{du}{dx} = 1$

=> $du = dx$

- We have a small problem in this question however, if we simply subbed in u and dx as we would still be stuck with an x above the line:

$\int_1^2 \frac{x}{x+1}.dx = \int_1^2 \frac{x}{u}.du$

- To sort this, we use our ★ to get x in terms of u:

$u = x + 1$

=> $x = u - 1$zW

- So our original integral now reduces to:

$\int_1^2 \frac{u-1}{u}.du$

$= \int_1^2 1 - \frac{1}{u}.du$ (Breaking up the fraction $\frac{u-1}{u}$ into $\frac{u}{u} - \frac{1}{u}$)

$= u - \ln u$

- Again, we have two options to finish, but we will go with subbing in for u here:

$u - \ln u = x + 1 - \ln(x + 1)$

$= 2 + 1 - \ln(2 + 1) - [1 + 1 - \ln(1 + 1)]$

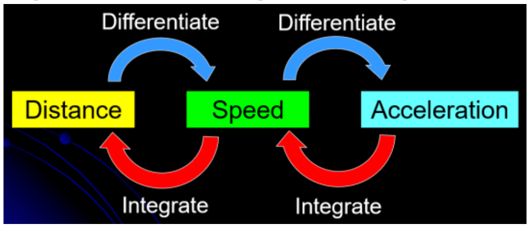
$= 3 - \ln 3 - 2 + \ln 2$

$= 1 + \ln\frac{2}{3}$

Day 2: Classwork Questions: Pg 172 Ex 9E Qs 11/14/15/18/21

Topic 41: Rates of Change

- You learned on your Maths course that the main application of Differentiation is to find out how quickly a quantity is changing i.e. it's rate of change.
- You also learned about the connection between distance, velocity(speed) and acceleration.
- As we now know that Integration is the reverse process of Differentiation, we can also update and complete our diagram connecting them together.



- If we are given any of the quantities above as a function, we can find any of the others by either differentiating or integrating.

- Example 1: Pg 175 Ex 9F Q3

A girl throws a ball straight up in the air. After t seconds, the ball's height, h, in metres, above the ground is given by $h(t) = 30t - 5t^2$. Find (i) the height of the ball after 1 second (ii) the speed at which the ball is rising at t = 1s (iii) the time when the ball reaches its maximum height.

Solution:

- In this question, we have been given the height of the ball, which is a distance function, so we can see where we are in the diagram above (the yellow box).
- A good idea is to begin by quickly differentiating once and then again to get expressions for speed and acceleration before starting the question:

Distance/Height = $30t - 5t^2$ m

Speed/Velocity = $30 - 10t$ m/s

Acceleration = -10 m/s^2

i) The height of the ball after 1 second can be found by subbing 1 into our height/distance function:

Height = $30(1) - 5(1)^2 = 25m$

ii) The speed of the ball at t = 1 can be found by subbing 1 into our speed function:

Speed = $30 - 10(1) = 20m/s$

iii) We learned in Projectiles that if a projectile reaches max height, then it's velocity at that point will be zero => Speed = 0

=> $30 - 10t = 0$

=> $t = 3secs$

Note: If we were asked for the max height itself, we could now substitute this time of 3 secs into our height function.

- Example 2: Pg 175 Ex 9F Q9

  A car is travelling so that its velocity (in m/s) at any time t is given by $v(t) = 2t + 50$.

  i) Show that it is accelerating uniformly.

  ii) What is its constant acceleration?

  iii) Use integration to find the distance it travels in the first 6 seconds.

  iv) How long does it take to travel 3600m?

  Solution :

  i) ii) In this example, we've been given the velocity function, so we are at the green box in our diagram above.

  - If we differentiate, we will get acceleration, and if we integrate, we will get distance:

  $v(t) = 2t + 50$

  => $a(t) =$ 2 m/s^2

  - So, the acceleration is a constant 2, which answers both parts (i) and (ii).

  iii) The question guides us as to how to proceed here, but we should know already that we need to integrate the velocity function to get distance:

  $v(t) = 2t + 50$

  => $s(t) = \int 2t + 50$

  => $s(t) = 2\left(\frac{t^2}{2}\right) + 50t$

  => $s(t) = t^2 + 50t + C$

  - We can assume here that the car had moved a distance of 0, when time started:

  => $0 = (0)^2 + 50(0) + C$

  => $C = 0$

  => $s(t) = t^2 + 50t$

  - We can now calculate how far the car travelled in 6 seconds:

  $s(6) = (6)^2 + 50(6) = 336m$

  iv) For this part, the distance travelled is 3600m:

  => $3600 = t^2 + 50t$

  => $t^2 + 50t - 3600 = 0$

  => $(t + 90)(t - 40) = 0$

  => $t = -90 \quad or \quad t = 40\ secs$

Day 1: Classwork Questions: Pg 175 Ex 9F Qs 2/5/8/10

Day 2: Classwork Questions: Pg 152 Ex 8C Qs 11/13/16 and then try Qs 19/21

- Topic 42: Differentiating Vectors

Can do Topics 42 and 42 in 1 lesson

- We can differentiate vectors also, by differentiating the $\vec{\imath}$ and $\vec{\jmath}$ parts separately.

- Example: Pg 178 Ex 9G Q3

The displacement vector (in metres) of a body from a fixed point O (at time t seconds) is given by $\vec{s} = (3t^3 - 1)\vec{\imath} + (100 - 4t^3)\vec{\jmath}$. (i) Find the displacement vector at t = 3s. (ii) Find the velocity vector at any time t. (iii) Find the speed of the body at time t = 3s. (iv) Find the acceleration vector at any time t. (v) Find the magnitude of the acceleration at t = 10s.

Solution:

i) For this part, we can just fill in the value for t:

$\vec{s} = (3t^3 - 1)\vec{\imath} + (100 - 4t^3)\vec{\jmath}$

=> @ t = 3: $\vec{s} = (3(3)^3 - 1)\vec{\imath} + (100 - 4(3)^3)\vec{\jmath}$

=> $\vec{s} = 80\vec{\imath} - 8\vec{\jmath}$

ii) To find velocity, we will have to differentiate the displacement vector:

$\vec{s} = (3t^3 - 1)\vec{\imath} + (100 - 4t^3)\vec{\jmath}$

=> $\vec{v} = 9t^2\vec{\imath} - 12t^2\vec{\jmath}$

iii) To find the speed at t = 3, we will first have to sub in t = 3 and then find the magnitude of the resulting vector:

$\vec{v} = 9t^2\vec{\imath} - 12t^2\vec{\jmath}$

=> $\vec{v} = 9(3)^2\vec{\imath} - 12(3)^2\vec{\jmath}$

=> $\vec{v} = 81\vec{\imath} - 108\vec{\jmath}$

=> Speed = $|81\vec{\imath} - 108\vec{\jmath}| = \sqrt{(81)^2 + (-108)^2} = 135\ m/s$

Answer @ back wrong?

iv) To find acceleration, we will have to differentiate the velocity vector:

$\vec{v} = 9t^2\vec{\imath} - 12t^2\vec{\jmath}$

=> $\vec{a} = 18t\vec{\imath} - 24t\vec{\jmath}$

v) First, we will sub in t = 10, and then get the magnitude:

$\vec{a} = 18t\vec{\imath} - 24t\vec{\jmath}$

=> $\vec{a} = 18(10)\vec{\imath} - 24(10)\vec{\jmath}$

=> $\vec{a} = 180\vec{\imath} - 240\vec{\jmath}$

=> Magn = $|180\vec{\imath} - 240\vec{\jmath}| = \sqrt{(180)^2 + (-240)^2} = 300$ m/s^2

Classwork Questions: Pg 178 Ex 9G Qs 2/4/5/7

Topic 43: Work done by a variable Force

- We saw in Chapter 5 that the work done (measured in Joules) by a fixed force over a distance s is given by: $\boxed{W = Fs}$
- We will now look at how to calculate work done, if the force is varying, which can be found using: $\boxed{W = \int_A^B F(x).dx}$

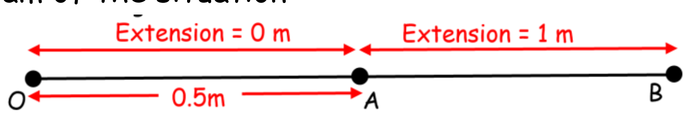
F(x) is the force, which is a function of distance x from some fixed point, between two points A and B

- **Example 1:** Pg 180 Ex 9H Q3

An elastic string has elastic constant 14 N/m and natural length 50 cm. (i) Find the work done by Alexa in stretching the string to a length of 1.5 m. (ii) If Alexa now stretches the string a further 1 m (so that it is 2.5 m long), what is the work done by her in this action? (iii) What potential energy does the string have now?

Solution:

i) Let's look at a diagram of the situation:



Method 1: Define 'x' as length of string.
From Hooke's Law, we know that the force exerted by a string is:

$$F = k(l - l_0) = 14(x - 0.5) = 14x - 7$$

- So, we can find the work done:

$$W = \int_A^B F(x).dx$$
$$=> W = \int_{0.5}^{1.5} 14x - 7.dx$$
$$=> W = 14\frac{x^2}{2} - 7x$$
$$=> W = 7x^2 - 7x$$

- We now sub in our limits:

$$W = 7(1.5)^2 - 7(1.5) - 7(0.5)^2 + 7(0.5) = 7J$$

N.B. Limits vary depending on how 'x' is defined

Method 2: Define 'x' as extension.
From Hooke's Law, we know that the force exerted by a string is:

$$F = kx = 14x$$

- So, we can find the work done:

$$W = \int_A^B F(x).dx$$
$$=> W = \int_0^1 14x.dx$$
$$=> W = 14\frac{x^2}{2}$$
$$=> W = 7x^2$$

- We now sub in our limits:

$$W = 7(1)^2 - 7(0)^2 = 7J$$

ii) If the string is stretched to 2.5 m long from 1.5 m long then the extension changes from 1 to 2, so we can do:

'x' as length of string	'x' as extension
$W = 7(2.5)^2 - 7(2.5) - 7(1.5)^2 + 7(1.5) = 21J$	$W = 7(2)^2 - 7(1)^2 = 21J$

iii) The potential energy gained will be equal to the work done in stretching the string from its natural length to its final length i.e. the extension goes from 0 to 2:

'x' as length of string	'x' as extension
$P.E.Gain = Work\ Done$ $= 7(2.5)^2 - 7(2.5) - 7(0.5)^2 + 7(0.5) = 28J$	$P.E.Gain = Work\ Done$ $= 7(2)^2 - 7(0)^2 = 28J$

Classwork Questions: Pg 180 Ex 9H Qs 2/4/6 and Day 2: Pg 180 Ex 7/9/10

Revision Questions and Test